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Pareto Models (Extreme Value Theory in Two Slides)

- ▶ **Pareto distribution**, $\mathbb{P}[X \leq x] = F(x) = 1 - x^{-\alpha}$, $x \geq 1$,
- ▶ reinsurance (Hagstroem, 1925), fire insurance (Beirlant et al., 1999, $\alpha \approx 1$), commercial property insurance (Biffis and Chavez, 2014, $\alpha \approx 1.25$), earthquakes (Sornette et al., 1996, $\alpha \in (0.8; 1.2)$), hurricanes (Hsieh, 1999, $\alpha \approx 1.5$), Nuclear Power Accidents (Hofert and Wüthrich, 2012, $\alpha \in (0.6; 0.7)$), cyber risk (Eling and Wirfs, 2019)
- ▶ α is the tail parameter, $\alpha \in (0, \infty)$
 - ▶ $X \sim \mathcal{P}(\alpha)$ has finite variance, finite mean $\iff \alpha \in (2, \infty)$
 - ▶ $X \sim \mathcal{P}(\alpha)$ has infinite variance, finite mean $\iff \alpha \in (1, 2]$
 - ▶ $X \sim \mathcal{P}(\alpha)$ has infinite mean $\iff \alpha \in (0, 1]$
- ▶ *“the distribution of economic loss from earthquakes follows an approximate Pareto law, with a very heavy tail, and that the tail exponent $\alpha = 1$ by no means is unreasonable”*, Ibragimov et al., 2009

Pareto Models (Extreme Value Theory in Two Slides)

▶ $F(x) = 1 - x^{-\alpha}$ or $\log[1 - F(x)] = -\alpha \log[x]$

since $\hat{F}(x) = \frac{1}{n} \sum_{i=1}^n \mathbf{1}(x_i \leq x)$

▶ then $\log \left[\frac{n-i}{n} \right] = -\alpha \log[x_i]$ if $x_1 \leq x_2 \leq \dots \leq x_n$

▶ Pareto plot is $\log \frac{n-i}{n}$ against $\log x_i$

▶ see e.g. Zajdenweber, 1996

Figure 1
Size Distribution of Losses in 1983

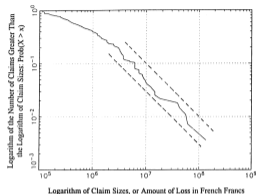
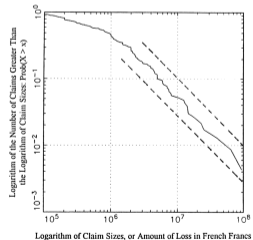
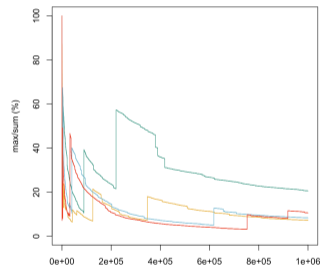
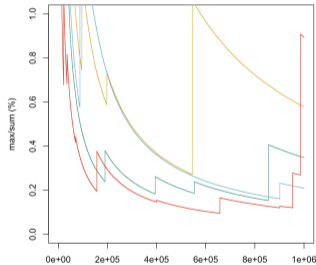
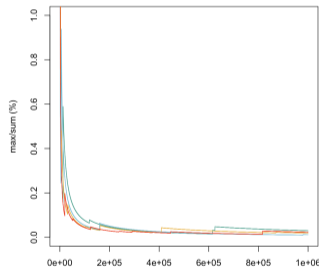


Figure 2
Size Distribution of Losses in 1986



Infinite mean ?

- Simulations of $n \mapsto \frac{\max\{x_1, \dots, x_n\}}{x_1 + \dots + x_n}$, Pareto $\alpha = 2.5$, $\alpha = 1.5$ and $\alpha = 0.95 < 1$.



Note first two, y scale [0%, 1%], third one [0%, 100%]

Insurance, Convex Order and Diversification

- ▶ Loss X (in \mathbb{R}_+), wealth w (in \mathbb{R}_+), utility u (increasing concave). $X \preceq Y$ if

$$\mathbb{E}[u(w - X)] \leq \mathbb{E}[u(w - Y)]$$

see Rothschild and Stiglitz, 1970

Def Consider two random variables X and Y , $X \preceq_{CX} Y$ if $\mathbb{E}[h(X)] \leq \mathbb{E}[h(Y)]$ for any convex function h

$\iff Y$ is a mean-preserving spread of X , i.e. $Y \stackrel{\mathcal{L}}{=} X + Z$, where $\mathbb{E}[Z|X] = 0$.

$\iff \mathbb{E}[(X - d)_+] \leq \mathbb{E}[(Y - d)_+]$ for all $d \in \mathbb{R}$.

$\implies \mathbb{E}[X] = \mathbb{E}[Y]$ and $\text{Var}[X] \preceq \text{Var}[Y]$.

Def Consider two random variables X and Y , $X \preceq_{ST} Y$ if $\mathbb{E}[h(X)] \leq \mathbb{E}[h(Y)]$ for any increasing function h

Insurance, Convex Order and Diversification

- ▶ Compare X_0 and $\mathbf{a}^\top \mathbf{X} = a_1 X_1 + \dots + a_n X_n$, with $a_i > 0$, $a_1 + \dots + a_n = 1$, and X_0, X_1, \dots, X_n are identically distributed.

- ▶ if X_i 's are random variables with finite mean,

$$\mathbf{a}^\top \mathbf{X} \preceq_{ST} X_0$$

- ▶ if X_i 's are independent random variables $\mathcal{P}(\alpha)$ with finite mean,

$$\mathbf{a}^\top \mathbf{X} \preceq_{ST} X_0$$

All risk-averse decision makers prefer the more diversified (Samuelson, 1967)

- ▶ if X_i 's are independent random variables $\mathcal{P}(\alpha)$ with infinite mean,

$$X_0 \preceq_{ST} \mathbf{a}^\top \mathbf{X}$$

All rational decision makers prefer the less diversified (see Ibragimov et al., 2009)

Non-diversification of Extreme Risks

▶ Suppose $\mathbf{a} = \left(\frac{1}{n}, \dots, \frac{1}{n}\right) = (n^{-1}, \dots, n^{-1})$, and consider (first) the case $\alpha = 1$

▶ Set $S(u_1, \dots, u_n) = \min_{i=1, \dots, n} \left\{ \frac{n}{i} u_{(i)} \right\}$ where $u_{(1)} \leq \dots \leq u_{(n)}$

▶ from Simes, 1986, if U_i 's are i.i.d $\mathcal{U}([0, 1])$, $S(U_1, \dots, U_n) \sim \mathcal{U}([0, 1])$

▶ since $u_1^{-1} + \dots + u_n^{-1} \geq j u_{(j)}^{-1}, \forall j, u_1^{-1} + \dots + u_n^{-1} \geq n(S(u_1, \dots, u_n))^{-1}$

$$\text{hence } \underbrace{U_1^{-1}}_{\mathcal{P}(1)} + \dots + \underbrace{U_n^{-1}}_{\mathcal{P}(1)} \geq n(S(U_1, \dots, U_n))^{-1} \stackrel{\mathcal{L}}{=} n \underbrace{U_1^{-1}}_{\mathcal{P}(1)}$$

▶ if $\alpha < 1$, Hardy et al., 1934,

$$\left(\frac{1}{n} (u_1^{-1/\alpha} + \dots + u_n^{-1/\alpha}) \right)^{-\alpha} \leq \left(\frac{1}{n} (u_1^{-1} + \dots + u_n^{-1}) \right)^{-1}$$

$$\text{hence } \underbrace{U_1^{-\alpha}}_{\mathcal{P}(\alpha)} + \dots + \underbrace{U_n^{-\alpha}}_{\mathcal{P}(\alpha)} \geq n(S(U_1, \dots, U_n))^{-\alpha} \stackrel{\mathcal{L}}{=} n \underbrace{U_1^{-\alpha}}_{\mathcal{P}(\alpha)}$$