Multi-Attribute Utility & Copulas

(based on Ali E. Abbas contributions)

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http://freakonometrics.hypotheses.org
Olivier’s Talk, part 2, on Independence & Additivity

Some fundamental concepts of multiattribute (expected) utility theory

- Utility independence
- Mutual utility independence
- Conditional utility functions
- Additive independence
- Scaling constants
- Additive/Multiplicative representation
Olivier’s Talk, part 2, on Utility Independence

Utility Independence

- Utility function $u(y, z)$
- $Y$ is utility independent of $Z$ when conditional preferences for lotteries on $Y$ given $z$ do not depend on the particular level of $z$
- Example: certainty equivalents does not depends on the value of $z^0$.

see also Keeney & Raiffa (1976)

**Definition.** We shall say that $Y$ is utility independent of $Z$ when conditional preferences for lotteries on $Y$ given $z$ do not depend on the particular level of $z$. 
Olivier’s Talk, part 2, on Mutual Utility Independence

Mutual Utility Independence

- When \( Y \) utility independent of \( Z \) and \( Z \) utility independent of \( Y \)
- \( Y \) and \( Z \) are said to be mutually utility independent
- Independence under certainty is a special case of mutual utility independence for degenerate lotteries
- when mutual utility independence holds
  - choose a conditional utility function \( u(., z^1) \)
  - choose a conditional utility function \( u(y^1, .) \)
  - choose the utility of an arbitrary consequence \((y^2, z^2)\)
  - allows to entirely specify \( u(x, y) \) for all \( y, z \).
- CAUTION: mutual utility independence does not imply an additive utility function.
Olivier’s Talk, part 2, on Additive Utility Independence

Additive Independence and additive utility function

- $Y$ and $Z$ are additive independent if and only if the two-attribute utility function is additive
- $u(y, z) = u(y, z^0) + u(y^0, z)$
- or $u(y, z) = k_Y u_Y(y) + k_Z u_Z(z)$ with
  - $u(y, z)$ normalized between $u(y^0, z^0) = 0$ and $u(y^1, z^1) = 1$
  - $u_Y(y)$ is a conditional utility function normalized between $u_Y(y^0) = 0$ and $u_Y(y^1) = 1$
  - $u_Z(z)$ is a conditional utility function normalized between $u_Z(z^0) = 0$ and $u_Z(z^1) = 1$
  - $k_Y = u(y^1, z^0)$ and $k_Z = u(y^0, z^1)$. 
Olivier’s Talk, part 2, on Additive Utility Independence

**Theorem 5.1.** Attributes $Y$ and $Z$ are additive independent if and only if the two-attribute utility function is additive. The additive form may be written either as

$$u(y, z) = u(y, z^o) + u(y^o, z), \quad (5.9)$$

or as

$$u(y, z) = k_Y u_Y(y) + k_Z u_Z(z) \quad (5.10)$$

where

1. $u(y, z)$ is normalized by $u(y^o, z^o) = 0$ and $u(y^1, z^1) = 1$ for arbitrary $y^1$ and $z^1$ such that $(y^1, z^o) > (y^o, z^o)$ and $(y^o, z^1) > (y^o, z^o)$.
2. $u_Y(y)$ is a conditional utility function on $Y$ normalized by $u_Y(y^o) = 0$ and $u_Y(y^1) = 1$.
3. $u_Z(z)$ is a conditional utility function on $Z$ normalized by $u_Z(z^o) = 0$ and $u_Z(z^1) = 1$.
4. $k_Y = u(y^1, z^o)$.
5. $k_Z = u(y^o, z^1)$. 

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Olivier’s Talk, part 2, on Mutual Utility Independence

The Multilinear Utility Function

- When $Y$ and $Z$ are said to be mutually utility independent, then $u(x, y)$ can be decomposed using the multilinear form:

$u(y, z) = k_Y u_Y(y) + k_Z u_Z(z) + k_{YZ} u_Y(y)u_Z(z)$  \hspace{1cm} (3)

or

$u(y, z) = u(y, z_0) + u(y_0, z) + k u(y, z_0)u(y_0, z)$  \hspace{1cm} (4)

- mutual utility independence reduces the dimensionality of the problem from 2 to $2 \times 1$

- $k_{YZ} = 1 - k_Y - k_Z$

- when $k_{YZ} \neq 0$ equivalent to a multiplicative representation

- when $k_{YZ} = 0$ equivalent to an additive representation

- scaling constants do not indicate the relative importance of attributes
Olivier’s Talk, part 2, on Mutual Utility Independence

**Theorem 5.2.** If $Y$ and $Z$ are mutually utility independent, then the two-attribute utility function is multilinear. In particular, $u$ can be written in the form

$$u(y, z) = u(y, z_0) + u(y_0, z) + ku(y, z_0)u(y_0, z), \tag{5.16}$$

or

$$u(y, z) = k_Y u_Y(y) + k_Z u_Z(z) + k_{YZ} u_Y(y) u_Z(z), \tag{5.17}$$

where

1. $u(y, z)$ is normalized by $u(y_0, z_0) = 0$ and $u(y_1, z_1) = 1$ for arbitrary $y_1$ and $z_1$ such that $(y_1, z_0) > (y_0, z_0)$ and $(y_0, z_1) > (y_0, z_0)$.
2. $u_Y(y)$ is a conditional utility function on $Y$ normalized by $u_Y(y_0) = 0$ and $u_Y(y_1) = 1$.
3. $u_Z(z)$ is a conditional utility function on $Z$ normalized by $u_Z(z_0) = 0$ and $u_Z(z_1) = 1$.
4. $k_Y = u(y_1, z_0)$.
5. $k_Z = u(y_0, z_1)$.
6. $k_{YZ} = 1 - k_Y - k_Z$ and $k = k_{YZ} / k_Y k_Z$. 
What are we looking for?

Ideally, we would like to obtain a representation of the utility function such that

$$u(x_1, x_2, \ldots, x_n) = f[f_1(x_1), f_2(x_2), \ldots, f_n(x_n)], \quad (5.5)$$

where $f_i$ is a function of attribute $X_i$ only, for $i = 1, 2, \ldots, n$, and where $f$ has a simple form, an additive or multiplicative form, for example.

See Sklar (1959) for cumulative distribution function for random vector $\mathbf{X} \in \mathbb{R}^n$,

$$F(x_1, \cdots, x_n) = C[F_1(x), \cdots, F_n(x_n)]$$

where $F(x) = \mathbb{P}[\mathbf{X} \leq x]$ and $F_i(x_i) = \mathbb{P}[X_i \leq x_i]$. 
What are we looking for?

Fig. 5.13. Different shapes of utility functions $u(y, z)$ where $Z$ is utility independent of $Y$. 
Historical Perspective

When everything else remains constant which do you prefer

\[(x_1, y_1) \text{ or } (x_2, y_2)\]

\(X\) can be consumption
\(Y\) can be health
(remaining life time expectancy)

Matheson & Howard (1968) : use a deterministic real-valued function \(V : \mathbb{R}^d \to \mathbb{R}\) and then use a utility function over the value function,

\[U(x) = U(x_1, \ldots, x_d) = u(V(x_1, \ldots, x_d)),\]

e.g. \(U(x) = u(x_1 + \cdots + x_d)\) or \(u(\min\{x_1, \ldots, x_d\})\).
Historical Perspective

See Matheson & Abbas (2005), e.g. $V(x, y) = xy^n$,

see also [Sheldon’s acoustic sweet spot](#) or peanut butter/jelly sandwich preference function
Historical Perspective

Alternative approach: assess utilities over individual attributes, and combine time into a functional form

Keeney & Raiffa (1976) : use some utility independence assumption

Mutual utility independence : \( U(x, y) = k_x u_x(x) + k_y u_y(y) + k_{xy} u_x(x) u_y(y) \)
where \( k_{xy} = 1 - k_x - k_y \)

Additive and Product forms

\( U(x, y) = k_x u_x(x) + k_y u_y(y) \) with \( k_x - k_y = 1 \)

\( U(x, y) = k_{xy} u_x(x) u_y(y) \)

Utility Independence is an interesting property, but it might be a simplifying one.
How to Construct Multi-Attribute Utility Functions

From Abbas & Howard (2005), in dimension $d = 2$,

$U(x, y) \in [0, 1]$ (normalization)

$U(x, y) = U(x, y) = 0$ (attribute dominance condition)
How to Construct Multi-Attribute Utility Functions

Non-decreasing with arguments:

- given $y, x_1 < x_2$ implies $(x_1, y) \preceq (x_2, y)$
- given $x, y_1 < y_2$ implies $(x, y_1) \preceq (x, y_2)$

$U(x, y) = u_x(x)$ and $U(x, y) = u_y(y)$
Conditional Utility

We can define conditional utility

\[ U_{y|x}(y|x) = \frac{U(x, y)}{u_x(x)} \]
**Conditional Utility**

Bayes’ Rule for Attribute Dominance Utility

\[ U(x, y) = u_x(x) \cdot U_{y|x}(y|x) = u_y(y) \cdot U_{x|y}(x|y). \]
Copula Structures for Attribute Dominance Utility

With two attributes, consider $U(x, y) = C(u_x(x), u_y(y))$

Since copulas are related to probability measures, function $C$ are 2-increasing. 

$C$ is the cumulative distribution function of some $U$, and

$$\mathbb{P}(U \in [a, b]) \geq 0$$

implies positive mixed partial derivatives, \( \frac{\partial^2 C(u,v)}{\partial u \partial v} \geq 0 \) (weaker condition exist).

Not a necessary condition for attribute dominance utility theory...
Understanding the Two Attribute Framework

$C$ might be on a normalized domain, with a normalized range $C : [0, 1]^2 \rightarrow [0, 1]$, with $C(0, 0) = 0$ and $C(1, 1) = 1$.

From Keeney & Raiffa (1976)

$X$ independent of $Y$ (preferences for lotteries over $x$ do not depend on $y$)

$$U(x, y) = k_2(y)U(x, y_0) + d_2(y)$$

$Y$ independent of $X$ (preferences for lotteries over $y$ do not depend on $x$)

$$U(x, y) = k_1(x)U(x_0, y) + d_1(x)$$

$C$ should satisfy some marginal property: there are $u_0$ and $v_0$ such that

$$C(u_0, v) = \alpha u_0 v + \beta u_0 \text{ and } C(u, v_0) = \alpha v_0 u + \beta v_0.$$ 

Margins are non decreasing, \( \frac{\partial C(u, v)}{\partial u} > 0 \) and \( \frac{\partial C(u, v)}{\partial v} > 0 \).
Understanding the Two Attribute Framework

Abbas & Howard (2005) defined some Class 1 Multiattribute Utility Copulas such that

\[ C(1, v) = \alpha_{u_0} v + \beta_{u_0} \] and \[ C(u, 1) = \alpha_{v_0} u + \beta_{v_0}. \]

**Proposition** Any multi-attribute utility function \( U(x_1, \cdots, x_n) \) that is continuous, bounded and strictly increasing in each argument can be expressed in terms of its marginal utility functions \( u_1(x_1), \cdots, u_n(x_n) \) and some class 1 multiattribute utility copula

\[ U(x_1, \cdots, x_n) = C[u_1(x_1), \cdots, u_n(x_n)]. \]
Archimedean Copulas

On probability cumulative distribution functions

\[ C(u_1, \cdots, u_d) = \phi^{-1}(\phi(u_1) + \cdots + \phi(u_d)) = \phi^{-1} \left( \sum_{j=1}^{n} \phi(u_j) \right) \]

with \( \phi : [0, 1] \to \mathbb{R}_+ \) an additive generator, or with \( \psi = \phi^{-1} \) completely monotone

\[ C(u_1, \cdots, u_d) = \psi(\psi^{-1}(u_1) + \cdots + \psi^{-1}(u_d)) = \psi \left( \sum_{j=1}^{n} \psi^{-1}(u_j) \right) \]

One can define some multiplicative generator, \( \lambda(t) = e^{-\phi(t)} \)

\[ C(u_1, \cdots, u_d) = \lambda^{-1}(\lambda(u_1) \times \cdots \times \lambda(u_d)) = \lambda^{-1} \left( \prod_{j=1}^{n} \lambda^{-1}(u_j) \right) \]

E.g. \( \phi(t) = -\log(t) \) or \( \lambda(t) = t \), independent copula, \( C = \Pi = C^\perp \)
Archimedean Utility Copulas

In the context of utility functions,

\[ C(v_1, \cdots, v_d) = \alpha \psi^{-1} \left( \prod_{i=1}^{d} \psi(\gamma_i + [1 - \gamma_i]v_i) \right) + [1 - \alpha] \]

with \( \gamma_i \in [0, 1] \), and such that \( a = \left[ \psi^{-1} \left( \prod_{i=1}^{d} \psi(\gamma_i) \right) \right]^{-1} \).

\( \psi \) continuous strictly increasing, \( \psi(0) = 0 \) and \( \psi(1) = 1 \).

E.g. \( \psi(t) = t \), then

\[ C(v_1, v_2) = \alpha [\gamma_1 + (1 - \gamma_1)v_1][\gamma_2 + (1 - \gamma_2)v_2] + (1 - \alpha) \]

i.e. multiplicative form of mutual independence.
Alternative to this Two Attribute Framework

By relaxing the condition of ‘attribute dominance’, Abbas & Howard (2005) defined some Class 2 Multiattribute Utility Copulas such that

\[ C(0, v) = \alpha_u v + \beta_u \] \quad and \quad \[ C(u, 0) = \alpha_v u + \beta_v. \]

Define a multiattribute utility copula \( C \) as a multivariate function of \( d \) variables satisfying \( C : [0, 1]^d \rightarrow [0, 1] \), with \( C(0) = 0 \), \( C(1) = 1 \), the following marginal property

\[ C(0, \cdots, 0, v_i, 0, \cdots, 0) = \alpha_i v_i + \beta_i, \] \quad with \( \alpha_i > 0 \)

and with \( \partial C(v)/\partial v_i > 0 \)
Alternative to this Two Attribute Framework

To define some Class 2 Archimedean utility copulas, let \( h \) be continuous on \([0, d]\), strictly increasing, with \( h(0) = 1 \) and \( h(1)^d \leq h(d) \). Then set

\[
C(v_1, \cdots, v_d) = \frac{h^{-1}\left(\prod_{j=1}^{d} h(\omega_j v_j)\right)}{h^{-1}\left(\prod_{j=1}^{d} h(\omega_j)\right)}, \text{ with } 0 \leq \omega_j \leq 1.
\]

E.g. \( h(t) = e^t \), then \( C(U_1(x_1), \cdots, U_d(x_d)) = \tilde{\omega}_1 U_1(x_1) + \cdots + \tilde{\omega}_d U_d(x_d) \), where \( \tilde{\omega}_j = \omega_j / (\omega_1 + \cdots + \omega_d) \), i.e. additive form of utility independence.
One-Switch Utility Independence

Introduced in Abbas & Bell (2011)

Consider two attributes \( x \) and \( y \), utility function \( U(x, y) \).

\( x \) is one-switch independent of \( y \) if and only if the ordering of any two lotteries over \( x \) switches at most once as \( y \) increases.

**Proposition** \( x \) is one-switch independent of \( y \) if and only if

\[
U(x, y) = g_0(y) + g_1(y)[f_1(x) + f_2(x) \cdot \varphi(y)]
\]

where \( g_1 \) has a constant sign, and \( \varphi \) is monotone.
One-Switch Utility Independence

\[ U(x, y) = g_0(y) + g_1(y)[f_1(x) + f_2(x)\varphi(y)] \]

It is possible to express those functions in terms of utility:

- \[ g_0(y) = U(x, y) \]
- \[ g_1(y) = [U(x, y) - U(x, y)] \]
- \[ f_1(x) = U(x|y) \]
- \[ f_2(x) = [U(x|y) - U(x|y)] \]

\[ \varphi(y) = \frac{U(x|y) - U(x|y)}{U(x|y) - U(x|y)} \]
Utility Trees and Bidirectional Utility Diagrams

From Abbas (2011), let \( \mathbf{x} = (x_i, x_{(i)}) \)

Consider the normalized conditional utility for \( x_i \) at \( \mathbf{x} \),

\[
U(x_i | \mathbf{x}_{(i)}) = \frac{U(x_i, \mathbf{x}_{(i)}) - U(x_i, x_{(i)})}{U(\overline{x_i}, \mathbf{x}_{(i)}) - U(\overline{x_i}, x_{(i)})}
\]

Note that

\[
U(x_i, \mathbf{x}_{(i)}) = U(\overline{x_i}, \mathbf{x}_{(i)}) \cdot U(x_i | \mathbf{x}_{(i)}) + U(x_i, \mathbf{x}_{(i)}) \cdot [1 - U(x_i | \mathbf{x}_{(i)})]
\]

Thus, for two attributes

\[
U(x, y) = U(\overline{x}, y) \cdot U(x | y) + U(x, y) \cdot [1 - U(x | y)]
\]
Utility Trees and Bidirectional Utility Diagrams

\[ U(x, y) = U(\bar{x}, y) \cdot U(x\mid y) + U(x, y) \cdot [1 - U(x\mid y)] \]

But it is also possible to expand it

\[ U(x, y) = \underbrace{U(\bar{x}, y)}_{=U(y\mid \bar{x}) \cdot U(\bar{x}, \bar{y})} \cdot U(x\mid y) + \underbrace{U(x, y)}_{=U(y\mid x) \cdot U(x, \bar{y})} \cdot [1 - U(x\mid y)] \]

which give four terms.

Simplified version can be obtained with additional assumptions:

Utility independence, \( U(x\mid y) = U(x\mid y) = U(x\mid \bar{y}) \ \forall y \)

Boundary independence, \( U(x\mid y) = U(x\mid \bar{y}) \)
Utility Trees and Bidirectional Utility Diagrams

\[ U(x, y) = U(\bar{x}, y) \cdot U(x|y) + U(x, y) \cdot [1 - U(x|y)] \]

\[ = U(y|x) \cdot U(x, y) + [1 - U(y|x)] \cdot U(x, y) \]

\[ U(x|y) \]
\[ (x^*, y^*) \]
\[ U(y|x^*) \]
\[ (x^*, y^0) \]
\[ \bar{U}(y|x^*) \]
\[ (x^0, y^0) \]
\[ (x, y) \sim \]
\[ U(x|y) \]
\[ (x^*, y) \sim \]
\[ \bar{U}(x|y) \]
\[ (x^0, y) \sim \]
\[ U(y|x^0) \]
\[ (x^0, y^*) \]
\[ \bar{U}(y|x^0) \]
\[ (x^0, y^0) \]
Utility Trees and Bidirectional Utility Diagrams

... and one can define directional utility diagrams

\[
x \quad \quad \quad y
\] : mutual utility independence

\[
x \rightarrow y
\] : Directional utility independence, \( x \) independent of \( y \)

\[
x \leftarrow y
\] : Directional utility independence, \( y \) independent of \( x \)

\[
x \leftrightarrow y
\] : no independence

In higher dimension, it is more complex...