

Archimax Copulas

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based on joint work with

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Agenda

- Copulas
- **Standard copula families**
- Elliptical distributions (and copulas)
- Archimedean copulas
- Extreme value distributions (and copulas)
- **Quantifying tail dependence**
- **Tails of Archimedean copulas**
- **Archimax copulas**
- Archimax copulas in dimension 2
- Archimax copulas in dimension $d \geq 3$

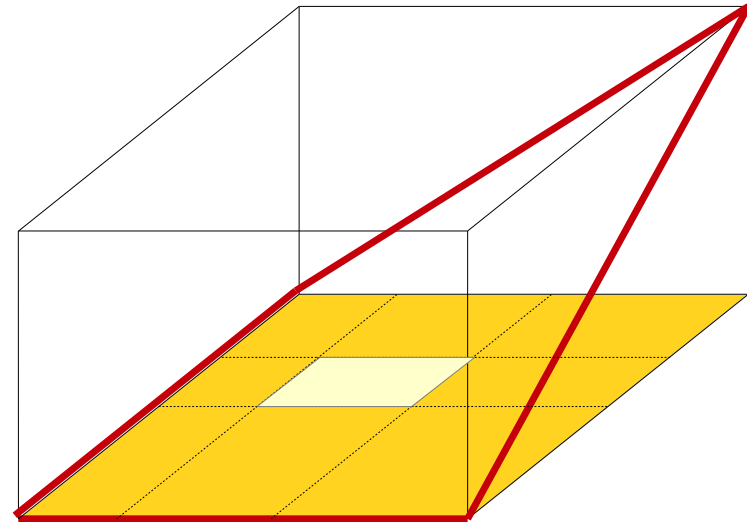
Copulas, in dimension $d = 2$

Definition 1

A **copula** in dimension 2 is a c.d.f on $[0, 1]^2$, with margins $\mathcal{U}([0, 1])$.

Thus, let $C(u, v) = \mathbb{P}(U \leq u, V \leq v)$,
where $0 \leq u, v \leq 1$, then

- $C(0, x) = C(x, 0) = 0 \quad \forall x \in [0, 1]$,
- $C(1, x) = C(x, 1) = x \quad \forall x \in [0, 1]$,
- and some *increasingness* property



Copulas, in dimension $d = 2$

Definition 2

A **copula** in dimension 2 is a c.d.f on $[0, 1]^2$, with margins $\mathcal{U}([0, 1])$.

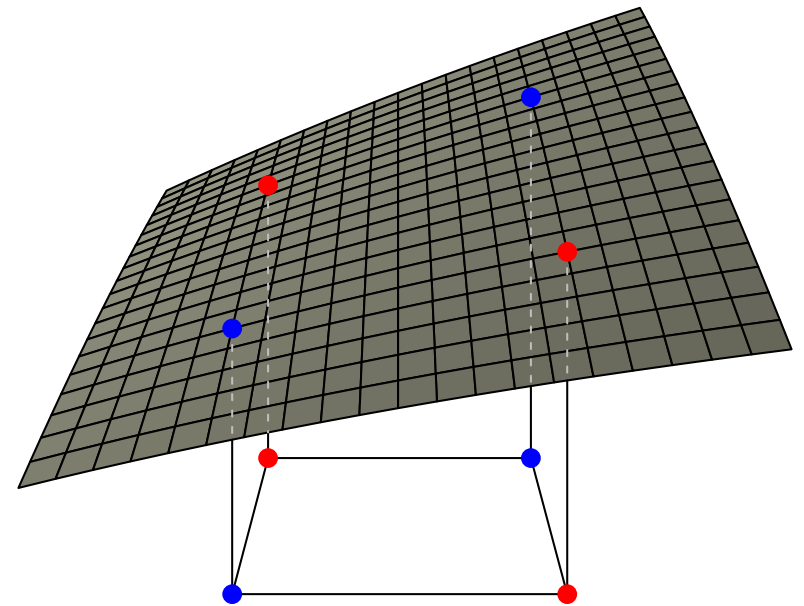
Thus, let $C(u, v) = \mathbb{P}(U \leq u, V \leq v)$,
where $0 \leq u, v \leq 1$, then

- $C(0, x) = C(x, 0) = 0 \quad \forall x \in [0, 1]$,
- $C(1, x) = C(x, 1) = x \quad \forall x \in [0, 1]$,
- If $0 \leq u_1 \leq u_2 \leq 1, 0 \leq v_1 \leq v_2 \leq 1$

$$C(u_2, v_2) + C(u_1, v_1) \geq C(u_1, v_2) + C(u_2, v_1)$$

(concept of 2-increasing function in \mathbb{R}^2)

see $C(u, v) = \int_0^v \int_0^u \underbrace{c(x, y)}_{\geq 0} dx dy$ with the density notation.



Copulas, in dimension $d \geq 2$

The concept of d -increasing function simply means that

$$\mathbb{P}(a_1 \leq U_1 \leq b_1, \dots, a_d \leq U_d \leq b_d) = \mathbb{P}(\mathbf{U} \in [\mathbf{a}, \mathbf{b}]) \geq 0$$

where $\mathbf{U} = (U_1, \dots, U_d) \sim C$ for all $\mathbf{a} \leq \mathbf{b}$ (where $a_i \leq b_i$).

Definition 3

Function $h : \mathbb{R}^d \rightarrow \mathbb{R}$ is d -increasing if for all rectangle $[\mathbf{a}, \mathbf{b}] \subset \mathbb{R}^d$, $V_h([\mathbf{a}, \mathbf{b}]) \geq 0$, where

$$V_h([\mathbf{a}, \mathbf{b}]) = \Delta_{\mathbf{a}}^{\mathbf{b}} h(\mathbf{t}) = \Delta_{a_d}^{b_d} \Delta_{a_{d-1}}^{b_{d-1}} \dots \Delta_{a_2}^{b_2} \Delta_{a_1}^{b_1} h(\mathbf{t}) \quad (1)$$

and for all \mathbf{t} , with

$$\Delta_{a_i}^{b_i} h(\mathbf{t}) = h(t_1, \dots, t_{i-1}, b_i, t_{i+1}, \dots, t_n) - h(t_1, \dots, t_{i-1}, a_i, t_{i+1}, \dots, t_n). \quad (2)$$

Copulas, in dimension $d \geq 2$

Definition 4

A copula in dimension d is a c.d.f on $[0, 1]^d$, with margins $\mathcal{U}([0, 1])$.

Theorem 1 1. If C is a copula, and F_1, \dots, F_d are univariate c.d.f., then

$$F(x_1, \dots, x_n) = C(F_1(x_1), \dots, F_d(x_d)) \quad \forall (x_1, \dots, x_d) \in \mathbb{R}^d \quad (3)$$

is a multivariate c.d.f. with $F \in \mathcal{F}(F_1, \dots, F_d)$.

2. Conversely, if $F \in \mathcal{F}(F_1, \dots, F_d)$, there exists a copula C satisfying (3). This copula is usually not unique, but it is if F_1, \dots, F_d are absolutely continuous, and then,

$$C(u_1, \dots, u_d) = F(F_1^{-1}(u_1), \dots, F_d^{-1}(u_d)), \quad \forall (u_1, \dots, u_d) \in [0, 1]^d \quad (4)$$

where quantile functions $F_1^{-1}, \dots, F_n^{-1}$ are generalized inverse (left cont.) of F_i 's.

If $\mathbf{X} \sim F$, then $\mathbf{U} = (F_1(X_1), \dots, F_d(X_d)) \sim C$.

Survival (or dual) copulas

Theorem 2 1. If C^* is a copula, and $\bar{F}_1, \dots, \bar{F}_d$ are univariate s.d.f., then

$$\bar{F}(x_1, \dots, x_d) = C^*(\bar{F}_1(x_1), \dots, \bar{F}_d(x_d)) \quad \forall (x_1, \dots, x_d) \in \mathbb{R}^d \quad (5)$$

is a multivariate s.d.f. with $F \in \mathcal{F}(F_1, \dots, F_d)$.

2. Conversely, if $F \in \mathcal{F}(F_1, \dots, F_d)$, there exists a copula C^* satisfying (5). This copula is usually not unique, but it is if F_1, \dots, F_d are absolutely continuous, and then,

$$C^*(u_1, \dots, u_d) = \bar{F}(\bar{F}_1^{-1}(u_1), \dots, \bar{F}_d^{-1}(u_d)), \quad \forall (u_1, \dots, u_d) \in [0, 1]^d \quad (6)$$

where quantile functions $F_1^{-1}, \dots, F_d^{-1}$ are generalized inverse (left cont.) of F_i 's.

If $\mathbf{X} \sim F$, then $\mathbf{U} = (F_1(X_1), \dots, F_d(X_d)) \sim C$ and $\mathbf{1} - \mathbf{U} \sim C^*$.

Benchmark copulas

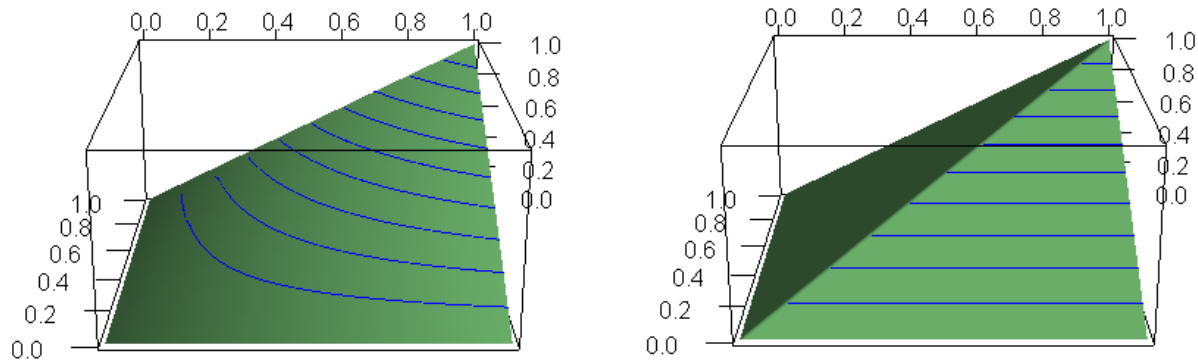
Definition 5

The independent copula C^\perp is defined as

$$C^\perp(u_1, \dots, u_d) = u_1 \times \dots \times u_d = \prod_{i=1}^d u_i.$$

Definition 6

The comonotonic copula C^+ (the Fréchet-Hoeffding upper bound of the set of copulas) is the copula defined as $C^+(u_1, \dots, u_d) = \min\{u_1, \dots, u_d\}$.



Spherical distributions

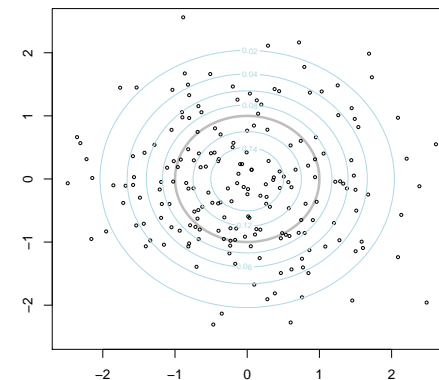
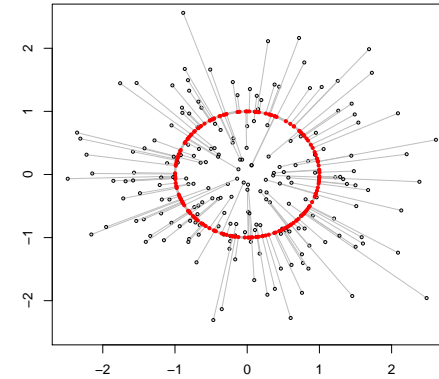
Definition 7

Random vector \mathbf{X} as a spherical distribution if

$$\mathbf{X} = R \cdot \mathbf{U}$$

where R is a positive random variable and \mathbf{U} is uniformly distributed on the unit sphere of \mathbb{R}^d , with $R \perp\!\!\!\perp \mathbf{U}$.

E.g. $\mathbf{X} \sim \mathcal{N}(\mathbf{0}, \mathbb{I})$.



Those distribution can be non-symmetric, see [HARTMAN & WINTNER \(AJM, 1940\)](#) or [CAMBANIS, HUANG & SIMONS \(JMVA, 1979\)](#)

Elliptical distributions

Definition 8

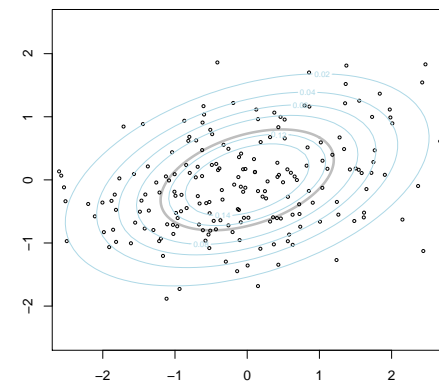
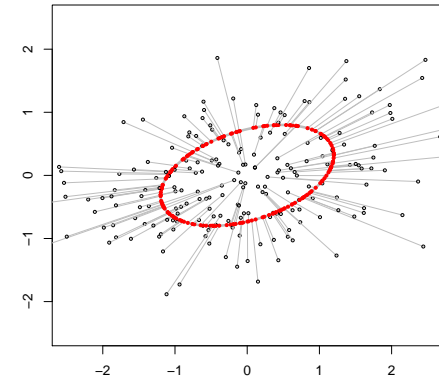
Random vector \mathbf{X} as a **elliptical distribution** if

$$\mathbf{X} = \boldsymbol{\mu} + R \cdot \mathbf{A} \cdot \mathbf{U}$$

where R is a positive random variable and \mathbf{U} is uniformly distributed on the unit sphere of \mathbb{R}^d , with $R \perp \mathbf{U}$, and where \mathbf{A} satisfies $\mathbf{A}\mathbf{A}' = \boldsymbol{\Sigma}$.

E.g. $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.

Elliptical distribution are popular in finance, see e.g. [JONDEAU, POON & ROCKINGER \(FMFM, 2008\)](#)



Archimedean copula

Definition 9

If $d \geq 2$, an Archimedean generator is a function $\phi : [0, 1] \rightarrow [0, \infty)$ such that ϕ^{-1} is d -completely monotone (i.e. ψ is d -completely monotone if ψ is continuous and $\forall k = 0, 1, \dots, d, (-1)^k d^k \psi(t) / dt^k \geq 0$).

Definition 10

Copula C is an Archimedean copula is, for some generator ϕ ,

$$C(u_1, \dots, u_d) = \phi^{-1}[\phi(u_1) + \dots + \phi(u_d)], \forall u_1, \dots, u_d \in [0, 1].$$

Exemple 1

$\phi(t) = -\log(t)$ yields the independent copula C^\perp .

$\phi(t) = [-\log(t)]^\theta$ yields Gumbel copula C_θ (note that $\psi(t) = \phi^{-1}(t) = \exp[-t^{1/\theta}]$).

Archimedean copula, exchangeability and frailties

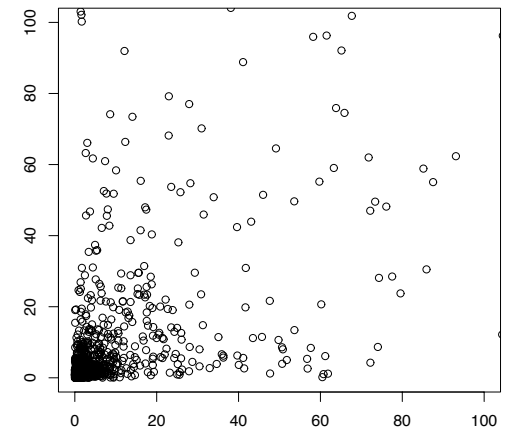
Consider residual life times $\mathbf{X} = (X_1, \dots, X_d)$ **conditionally independent** given some latent factor Θ , and such that $\mathbb{P}(X_i > x_i | \Theta = \theta) = \bar{B}_i(x_i)^\theta$. Then

$$\bar{F}(\mathbf{x}) = \mathbb{P}(\mathbf{X} > \mathbf{x}) = \psi \left(- \sum_{i=1}^n \log \bar{F}_i(x_i) \right)$$

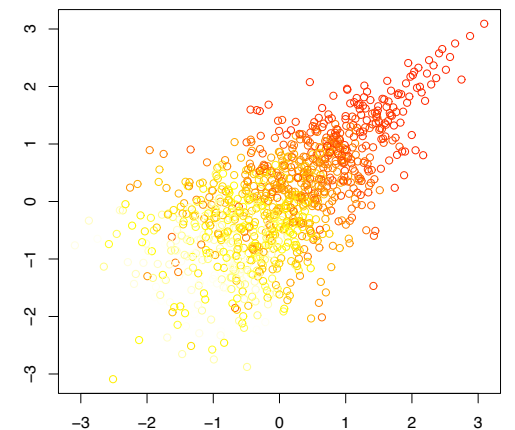
where ψ is the Laplace transform of Θ , $\psi(t) = \mathbb{E}(e^{-t\Theta})$. Thus, the survival copula of \mathbf{X} is Archimedean, with generator $\phi = \psi^{-1}$.

See [OAKES \(JASA, 1989\)](#).

Conditional independence, continuous risk factor



Conditional independence, continuous risk factor



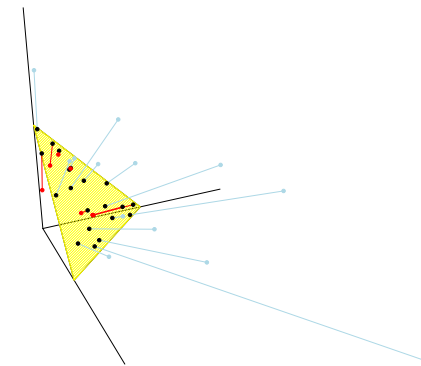
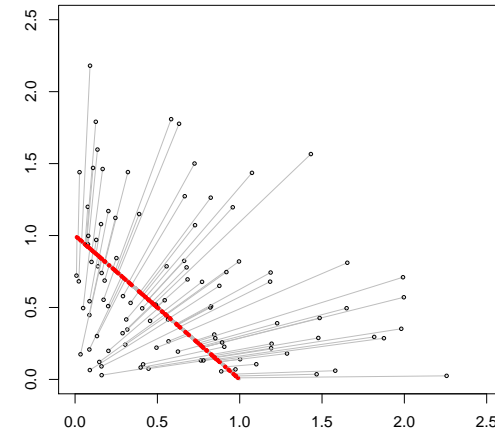
Stochastic representation of Archimedean copulas

Consider some strictly positive random variable R independent of \mathbf{U} , uniform on the simplex of \mathbb{R}^d . The survival copula of $\mathbf{X} = R \cdot \mathbf{U}$ is Archimedean, and its generator is the inverse of Williamson d -transform,

$$\phi^{-1}(t) = \int_x^\infty \left(1 - \frac{x}{t}\right)^{d-1} dF_R(t).$$

Note that $R \stackrel{\mathcal{L}}{=} \phi(U_1) + \dots + \phi(U_d)$.

See NEŠLEHOVÁ & MCNEIL (AS, 2009).



Archimedean copula and distortion

Definition 11

Function $h : [0, 1] \rightarrow [0, 1]$ defined as $h(t) = \exp[-\phi(t)]$ is called a **distortion function**.

GENEST & RIVEST (SPL, 2001), MORILLAS (M, 2005) considered distorted copulas (also called *multivariate probability integral transformation*)

Definition 12

Let h be some distortion function, and C a copula, then

$$C_h(u_1, \dots, u_d) = h^{-1}(C(h(u_1), \dots, h(u_d)))$$

is a copula.

Exemple2

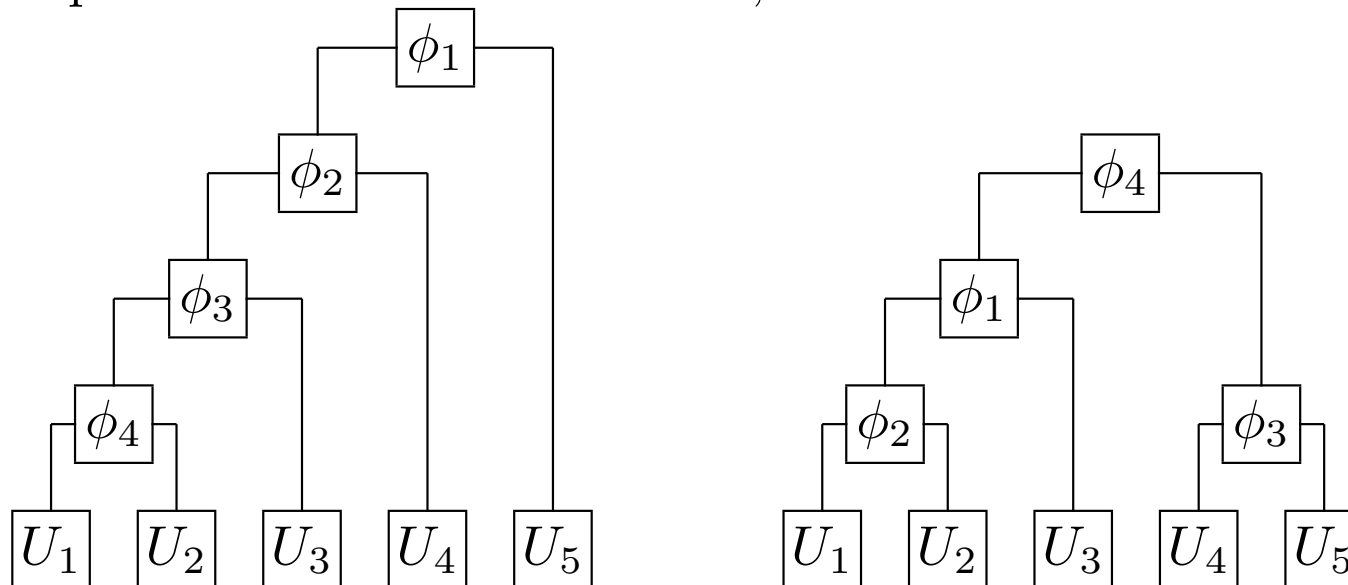
If $C = C^\perp$, then C_h^\perp is the Archimedean copula with generator $\phi(t) = -\log h(t)$.

Nested Archimedean copula, and hierarchical structures

Consider $C(u_1, \dots, u_d)$ defined as

$$\phi_1^{-1}[\phi_1[\phi_2^{-1}(\phi_2[\dots\phi_{d-1}^{-1}[\phi_{d-1}(u_1) + \phi_{d-1}(u_2)] + \dots + \phi_2(u_{d-1}))] + \phi_1(u_d)]]$$

where ϕ_i 's are generators. Then C is a copula if $\phi_i \circ \phi_{i-1}^{-1}$ is the inverse of a Laplace transform, and is called **fully nested Archimedean copula**. Note that partial nested copulas can also be considered,



(Univariate) extreme value distributions

Central limit theorem, $X_i \sim F$ i.i.d. $\frac{\bar{X}_n - b_n}{a_n} \xrightarrow{\mathcal{L}} S$ as $n \rightarrow \infty$ where S is a non-degenerate random variable.

Fisher-Tippett theorem, $X_i \sim F$ i.i.d., $\frac{X_{n:n} - b_n}{a_n} \xrightarrow{\mathcal{L}} M$ as $n \rightarrow \infty$ where M is a non-degenerate random variable.

Then

$$\mathbb{P} \left(\frac{X_{n:n} - b_n}{a_n} \leq x \right) = F^n(a_n x + b_n) \rightarrow G(x) \text{ as } n \rightarrow \infty, \forall x \in \mathbb{R}$$

i.e. F belongs to the max domain of attraction of G , G being an **extreme value distribution**: the limiting distribution of the normalized maxima.

$$-\log G(x) = (1 + \xi x)_+^{-1/\xi}$$

(Multivariate) extreme value distributions

Assume that $\mathbf{X}_i \sim F$ i.i.d.,

$$F^n(\mathbf{a}_n \mathbf{x} + \mathbf{b}_n) \rightarrow G(\mathbf{x}) \text{ as } n \rightarrow \infty, \forall \mathbf{x} \in \mathbb{R}^d$$

i.e. F belongs to the max domain of attraction of G , G being an (multivariate) extreme value distribution : the limiting distribution of the normalized componentwise maxima,

$$\mathbf{X}_{n:n} = (\max\{X_{1,i}\}, \dots, \max\{X_{d,i}\})$$

$$-\log G(\mathbf{x}) = \mu([\mathbf{0}, \infty) \setminus [\mathbf{0}, \mathbf{x}]), \forall \mathbf{x} \in \mathbb{R}_+^d$$

where μ is the exponent measure. It is more common to use the stable tail dependence function ℓ defined as

$$\ell(\mathbf{x}) = \mu([\mathbf{0}, \infty) \setminus [\mathbf{0}, \mathbf{x}^{-1}]), \forall \mathbf{x} \in \mathbb{R}_+^d$$

i.e.

$$-\log G(\mathbf{x}) = \ell(-\log G_1(x_1), \dots, \log G_d(x_d)), \quad \forall \mathbf{x} \in \mathbb{R}^d$$

Note that there exists a finite measure H on the simplex of \mathbb{R}^d such that

$$\ell(x_1, \dots, x_d) = \int_{\mathcal{S}_d} \max\{\omega_1 x_1, \dots, \omega_d x_d\} dH(\omega_1, \dots, \omega_d)$$

for all $(x_1, \dots, x_d) \in \mathbb{R}_+^d$, and $\int_{\mathcal{S}_d} \omega_i dH(\omega_1, \dots, \omega_d) = 1$ for all $i = 1, \dots, n$.

Definition 13

Copula $C : [0, 1]^d \rightarrow [0, 1]$ is an **multivariate extreme value copula** if and only if there exists a **stable tail dependence function** such that ℓ

$$C_\ell(u_1, \dots, u_d) = \exp[-\ell(-\log u_1, \dots, -\log u_d)]$$

Assume that $U_i \sim \Gamma$ i.i.d.,

$$\Gamma^n(\mathbf{u}^{\frac{1}{n}}) = \Gamma^n(u_1^{\frac{1}{n}}, \dots, u_d^{\frac{1}{n}}) \rightarrow C_\ell(\mathbf{u}) \text{ as } n \rightarrow \infty, \quad \forall \mathbf{x} \in \mathbb{R}^d$$

i.e. Γ belongs to the max domain of attraction of C_ℓ , C_ℓ being an **(multivariate) extreme value copula**, $\Gamma \in \text{MDA}(C_\ell)$.

The stable tail dependence function $\ell(\cdot)$

Observe that

$$n \left[1 - C \left(1 - \frac{x_1}{n}, \dots, 1 - \frac{x_d}{n} \right) \right] \rightarrow \underbrace{-\log [\Gamma(e^{-x_1}, \dots, e^{-x_d})]}_{=\ell(\mathbf{x})}$$

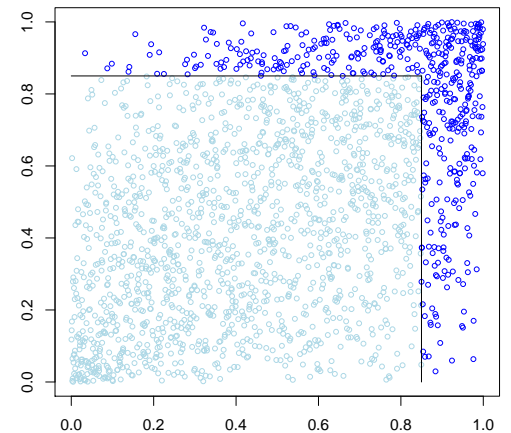
Exemple3

Gumbel copula, $\theta \in [1, +\infty]$,

$$\ell_\theta(x_1, \dots, x_d) = (x_1^\theta + \dots + x_d^\theta)^{1/\theta} = \|\mathbf{x}\|_\theta \quad \forall \mathbf{x} \in \mathbb{R}_+^d$$

Function $\ell(\cdot)$ satisfies

$$\underbrace{\max\{x_1, \dots, x_d\}}_{\substack{\text{(asympt.) comonotonicity} \\ \ell_\infty(\mathbf{x})}} \leq \ell(\mathbf{x}) \leq \underbrace{x_1 + \dots + x_d}_{\substack{\text{(asympt.) independence} \\ \ell_1(\mathbf{x})}}$$



The stable tail dependence function $\ell(\cdot)$

Function $\ell(\cdot)$ is homogeneous, $\ell(t \cdot \mathbf{x}) = t \cdot \ell(\mathbf{x}) \forall t \in \mathbb{R}_+$.

—→ consider the restriction of $\ell(\cdot)$ on the unit simplex Δ_{d-1} ,

$$\ell(\mathbf{x}) = \|\mathbf{x}\|_1 \cdot \underbrace{\ell\left(\frac{x_1}{\|\mathbf{x}\|}, \dots, \frac{x_d}{\|\mathbf{x}\|}\right)}_{\ell(\boldsymbol{\omega})} = \|\mathbf{x}\|_1 \cdot A(\omega_1, \dots, \omega_{d-1})$$

where $A(\cdot)$ is **Pickands dependence function**. Observe that

$$\max\{\omega_1, \dots, \omega_{d-1}, \omega_d\} \leq A(\omega_1, \dots, \omega_{d-1}) \leq 1, \quad \forall \boldsymbol{\omega} \in \Delta_{d-1}$$

(see BEIRLANT, GOEGEBEUR, SEGERS & TEUGELS (2004)).

Strong tail dependence

Definition 14

Let (X, Y) denote a random pair, the upper and lower tail dependence parameters are defined, if the limit exist, as

$$\begin{aligned}\lambda_L &= \lim_{u \rightarrow 0} \mathbb{P} (X \leq F_X^{-1}(u) | Y \leq F_Y^{-1}(u)) , \\ &= \lim_{u \rightarrow 0} \mathbb{P} (U \leq u | V \leq u) = \lim_{u \rightarrow 0} \frac{C(u, u)}{u},\end{aligned}$$

and

$$\begin{aligned}\lambda_U &= \lim_{u \rightarrow 1} \mathbb{P} (X > F_X^{-1}(u) | Y > F_Y^{-1}(u)) \\ &= \lim_{u \rightarrow 0} \mathbb{P} (U > 1 - u | V \leq 1 - u) = \lim_{u \rightarrow 0} \frac{C^*(u, u)}{u}.\end{aligned}$$

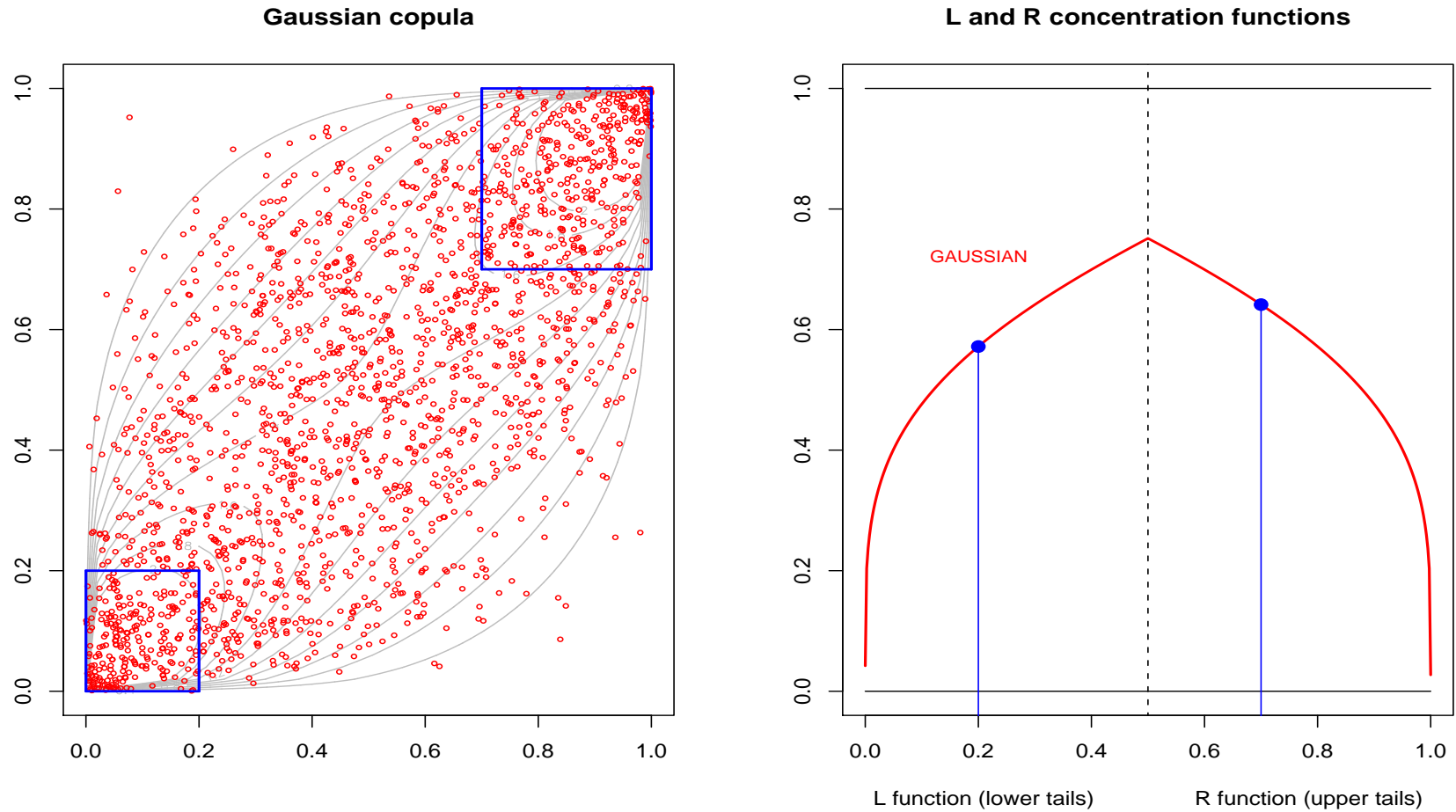


Figure 1: L and R cumulative curves.

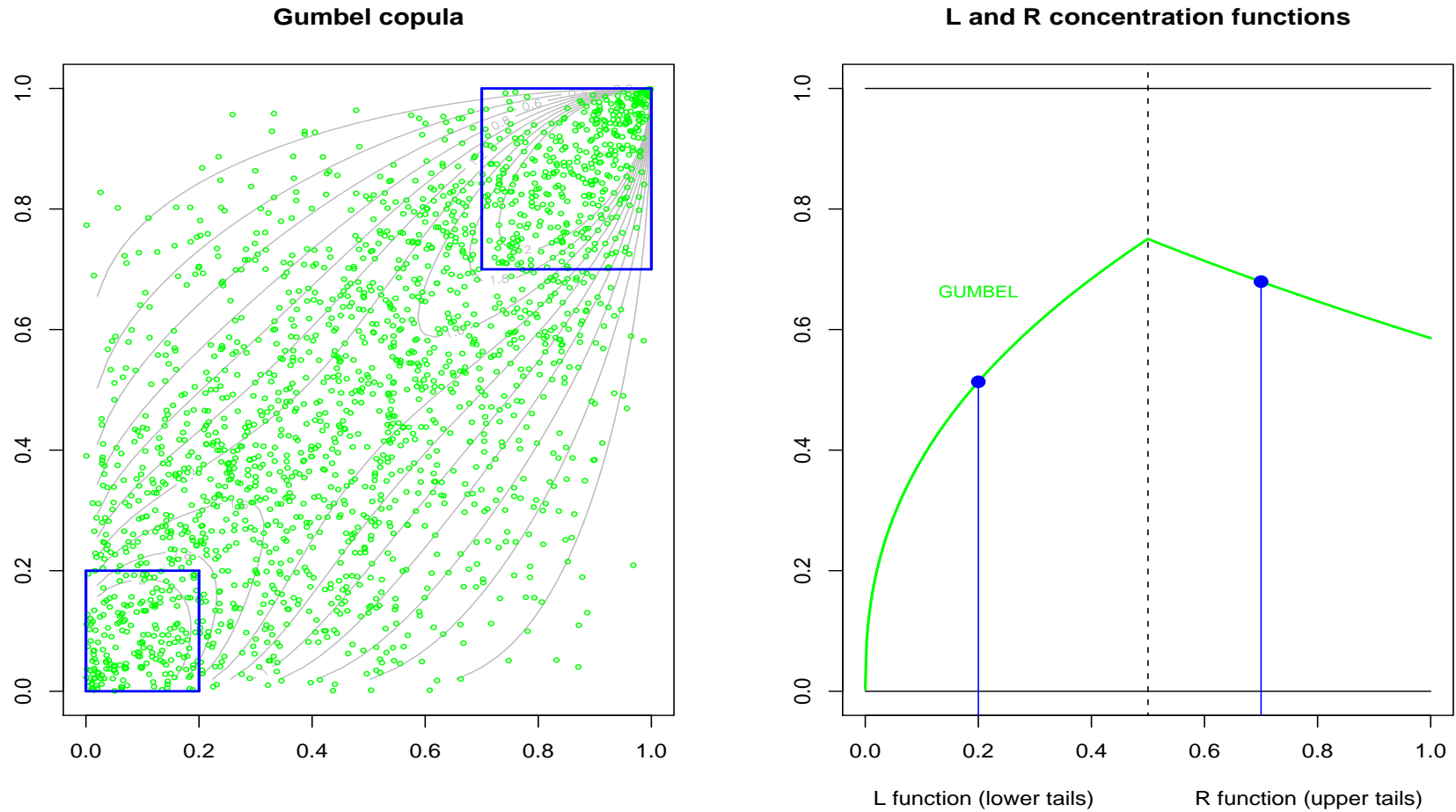


Figure 2: L and R cumulative curves.

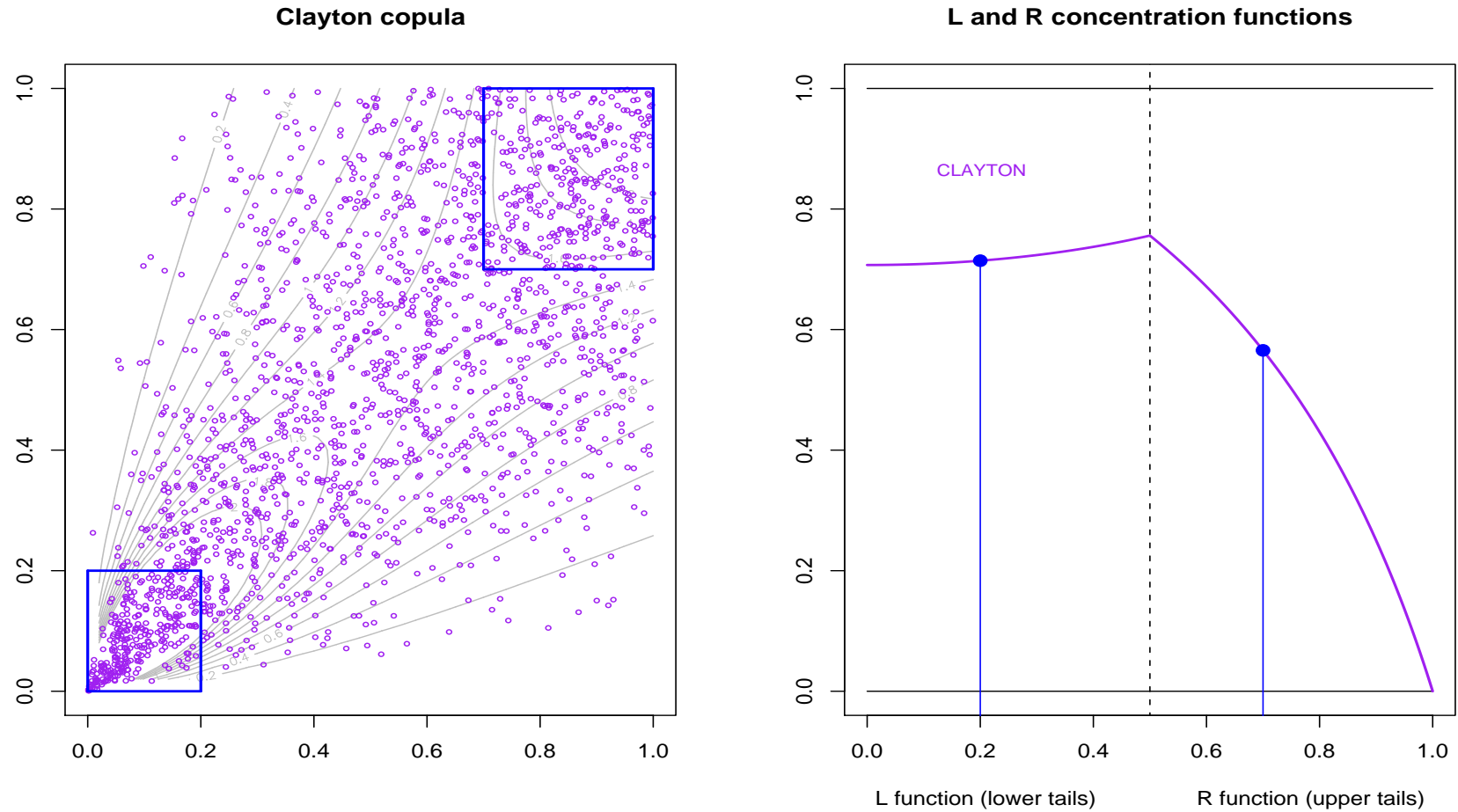


Figure 3: L and R cumulative curves.

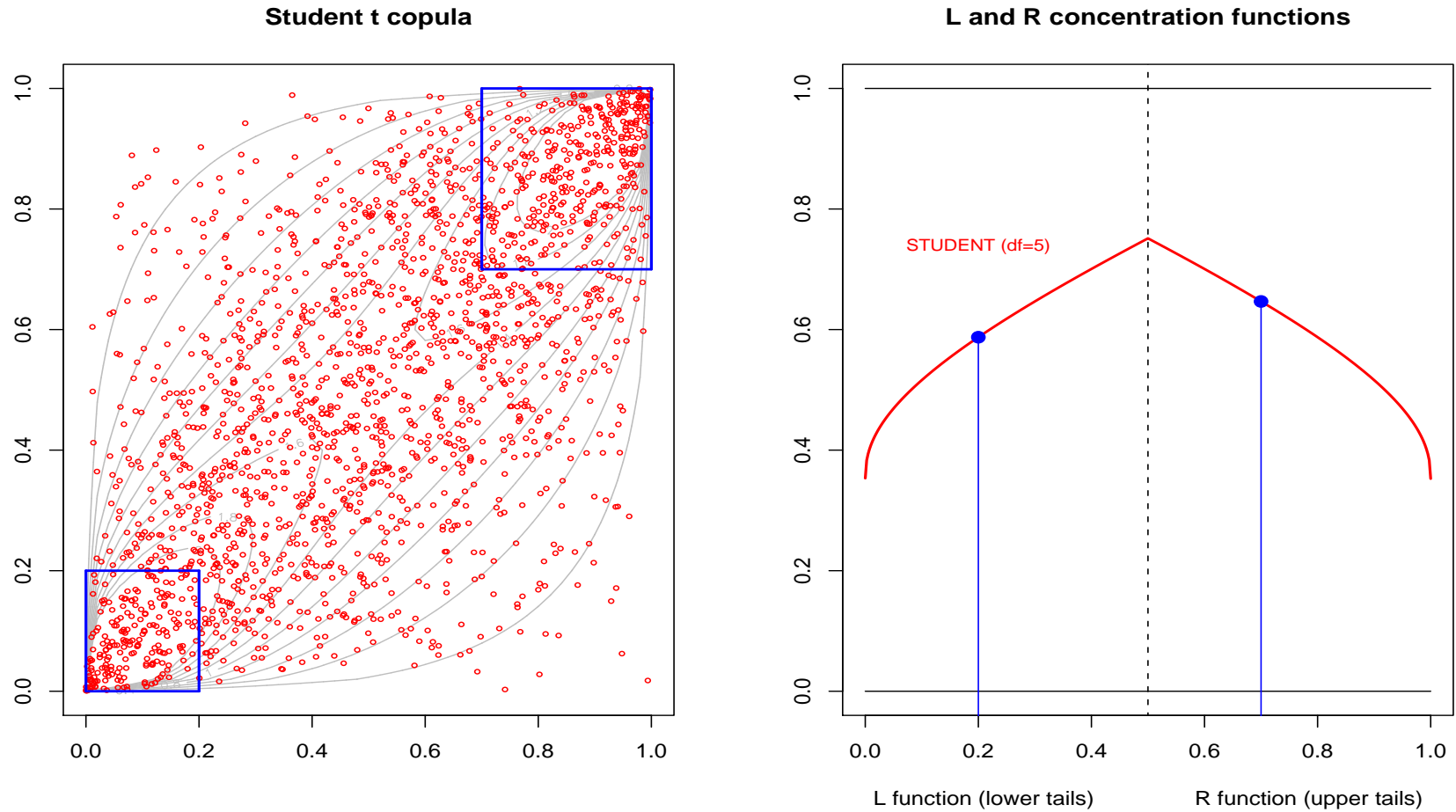


Figure 4: L and R cumulative curves.

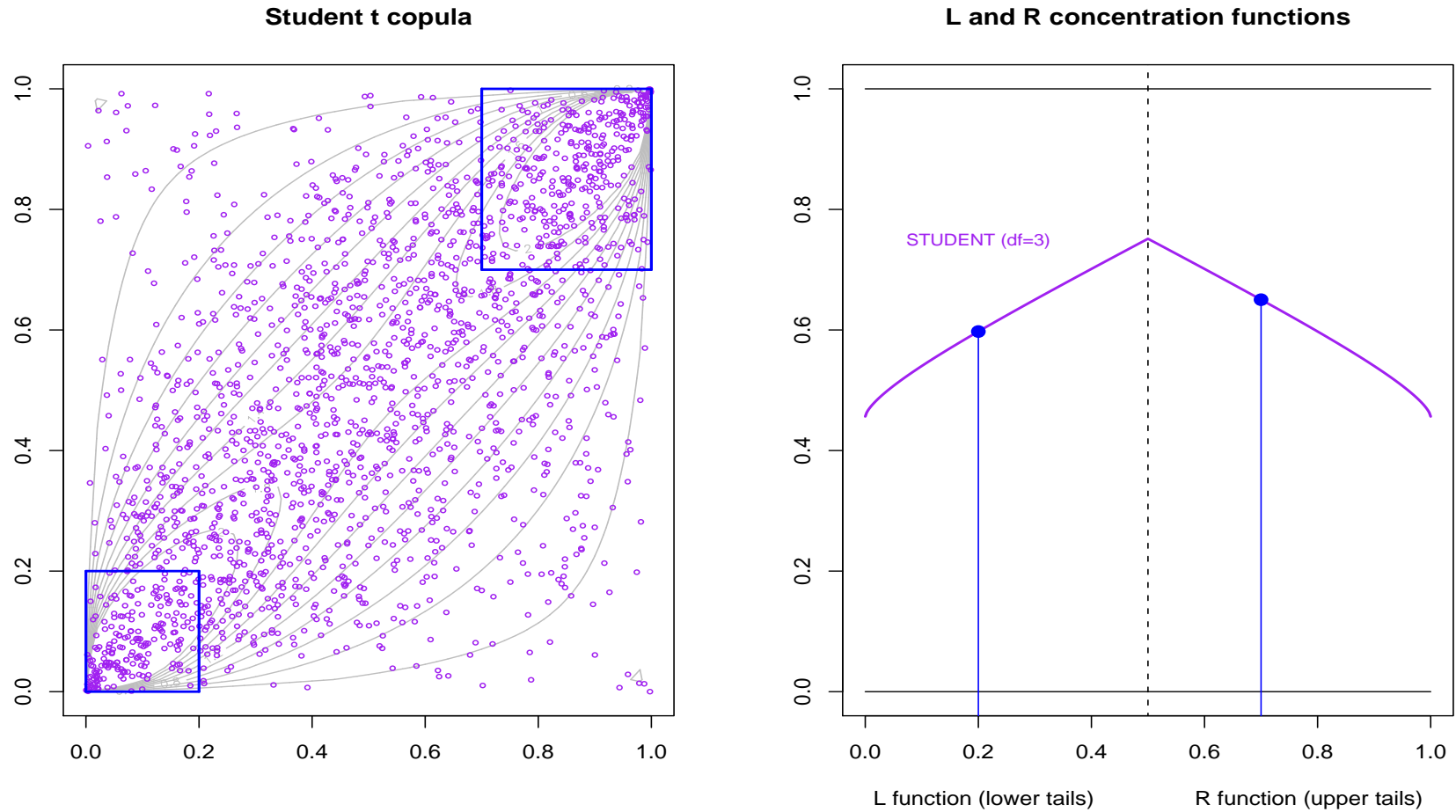


Figure 5: L and R cumulative curves.

Weak tail dependence

If Z_1 and Z_2 are **independent** (in tails), for u large enough

$$\mathbb{P}(Z_1 > F_1^{-1}(u), Z_2 > F_2^{-1}(u)) = \mathbb{P}(Z_1 > F_1^{-1}(u)) \cdot \mathbb{P}(Z_2 > F_2^{-1}(u)) = (1 - u)^2,$$

or equivalently, $\log \mathbb{P}(Z_1 > F_1^{-1}(u), Z_2 > F_2^{-1}(u)) = 2 \log(1 - u)$,

If Z_1 are **comonotonic** (in tails), for u large enough

$$\mathbb{P}(Z_1 > F_1^{-1}(u), Z_2 > F_2^{-1}(u)) = \mathbb{P}(Z_1 > F_1^{-1}(u)) = (1 - u)^1,$$

or equivalently, $\log \mathbb{P}(Z_1 > F_1^{-1}(u), Z_2 > F_2^{-1}(u)) = 1 \log(1 - u)$.

\implies limit of the ratio $\frac{\log(1 - u)}{\log \mathbb{P}(Z_1 > F_1^{-1}(u), Z_2 > F_2^{-1}(u))}$.

Weak tail dependence

COLES, HEFFERNAN & TAWN (E, 1999) defined

Definition 15

Let (X, Y) denote a random pair, the upper and lower tail dependence parameters are defined, if the limit exist, as

$$\eta_L = \lim_{u \rightarrow 0} \frac{\log(u)}{\log \mathbb{P}(Z_1 \leq F_1^{-1}(u), Z_2 \leq F_2^{-1}(u))} = \lim_{u \rightarrow 0} \frac{\log(u)}{\log C(u, u)},$$

and

$$\eta_U = \lim_{u \rightarrow 1} \frac{\log(1 - u)}{\log \mathbb{P}(Z_1 > F_1^{-1}(u), Z_2 > F_2^{-1}(u))} = \lim_{u \rightarrow 0} \frac{\log(u)}{\log C^*(u, u)}.$$

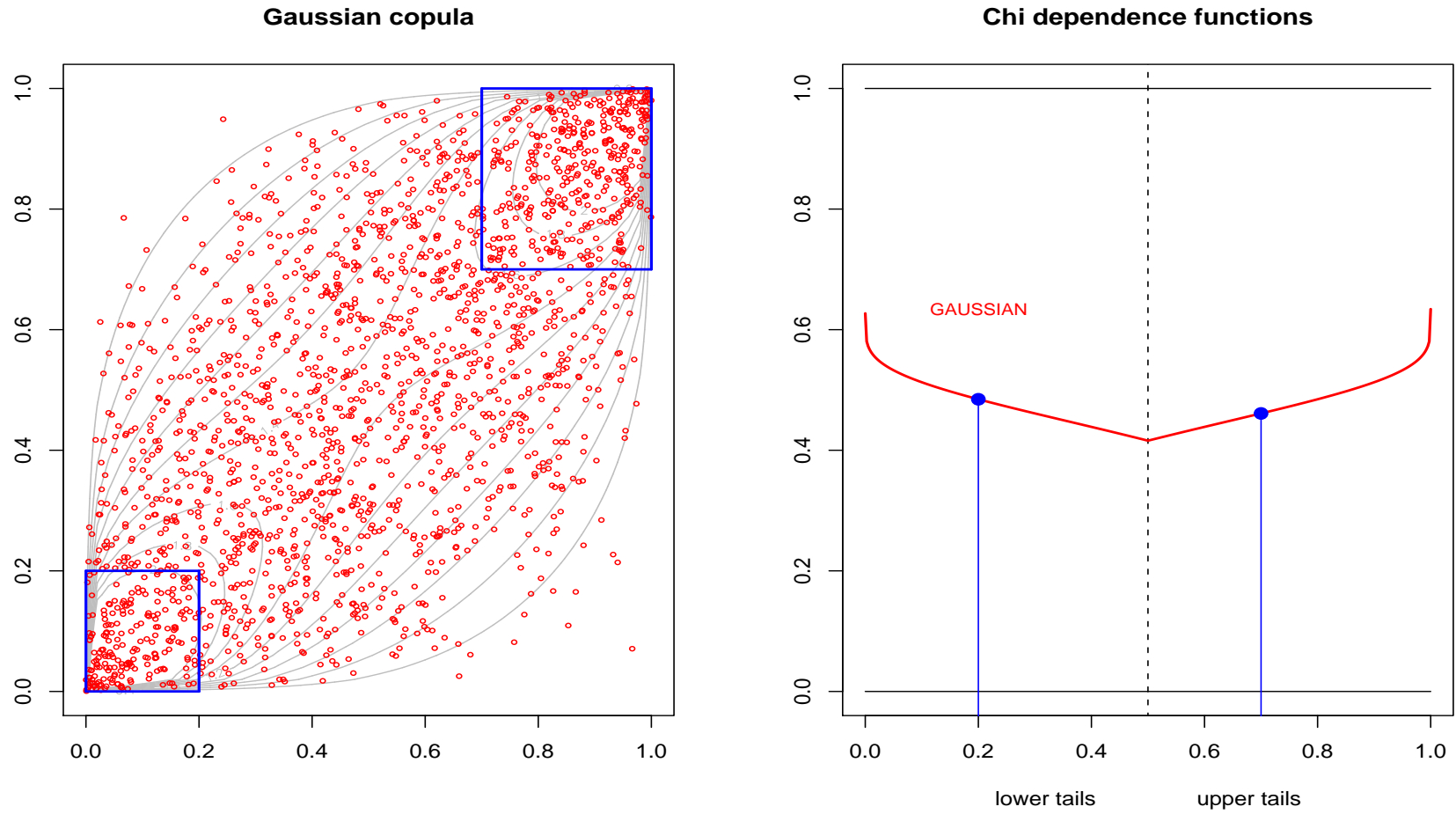


Figure 6: $\bar{\chi}$ functions.

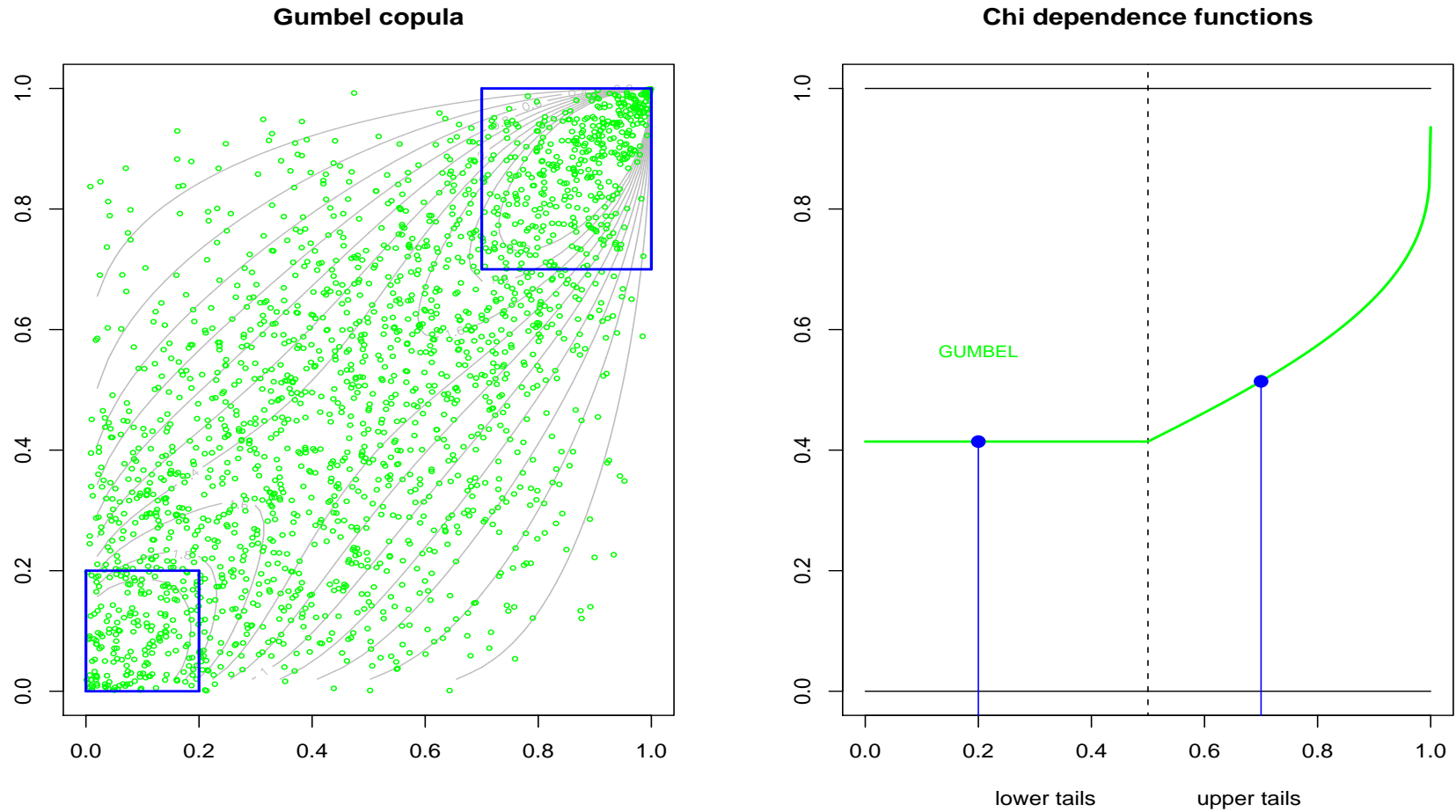


Figure 7: $\bar{\chi}$ functions.

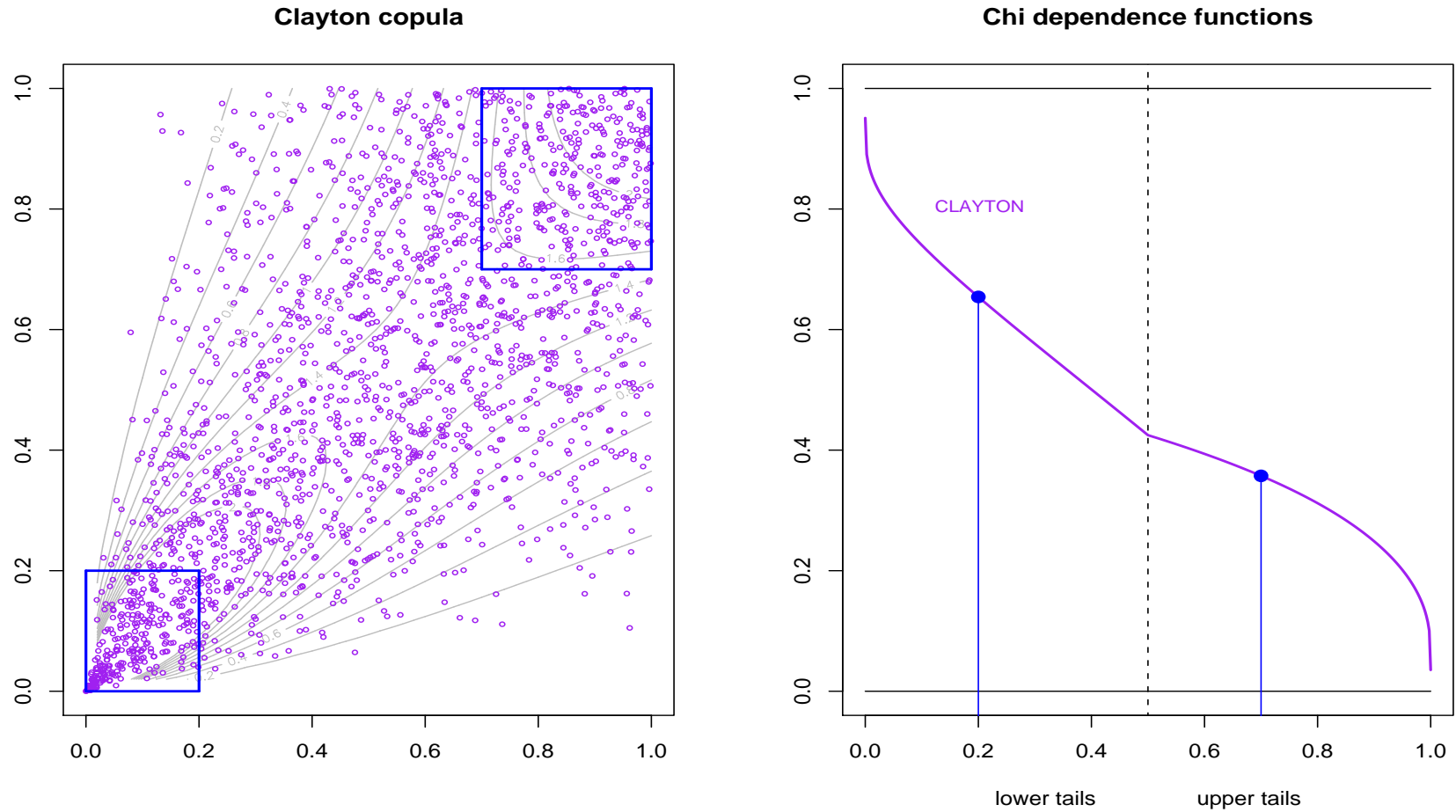


Figure 8: $\bar{\chi}$ functions.

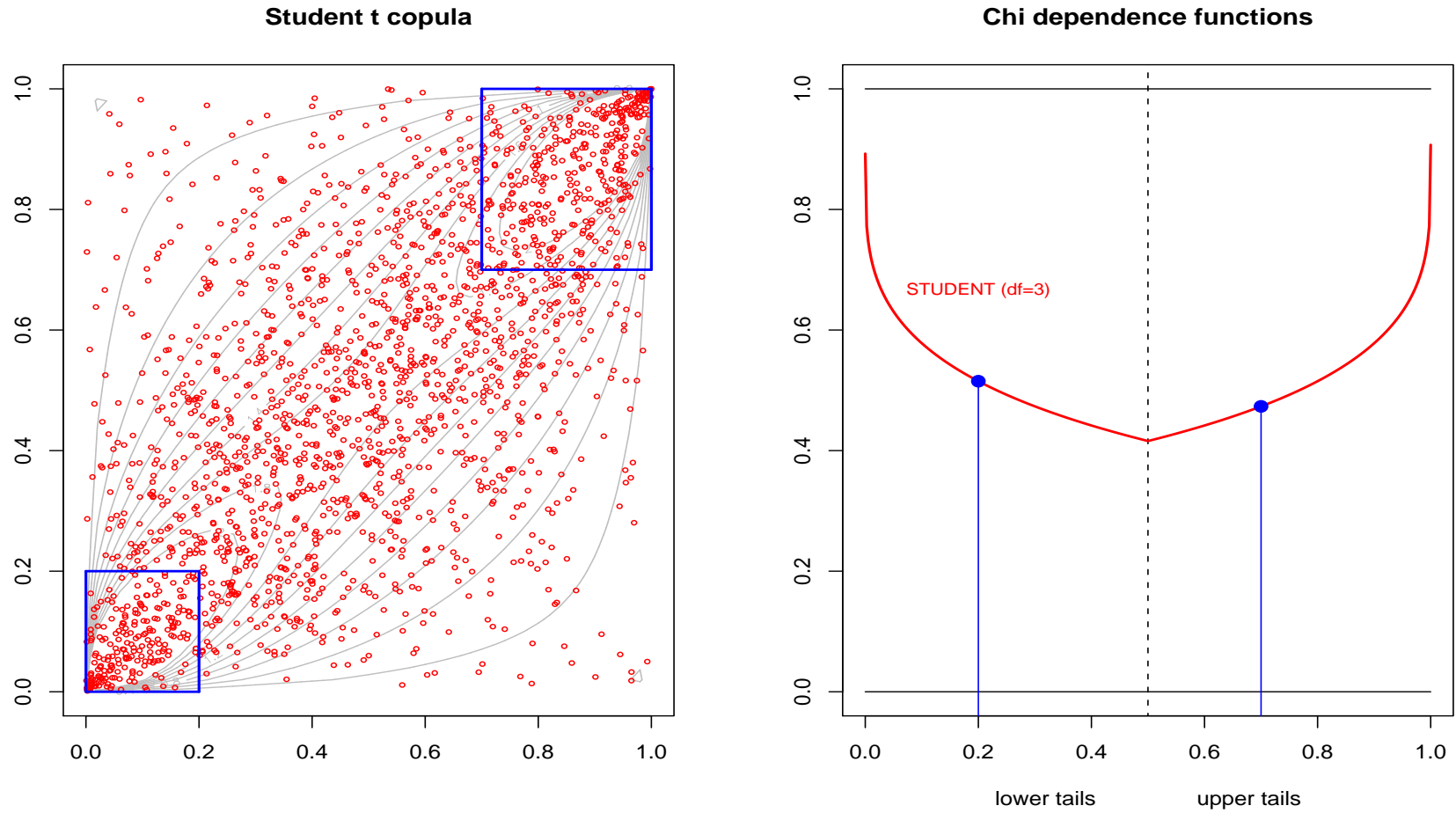


Figure 9: $\bar{\chi}$ functions.

Lower tails of Archimedean copulas

Study regular variation property of ϕ at 0,

$$\lim_{s \rightarrow 0} \frac{\phi(st)}{\phi(s)} = t^{-\theta_0}, t \in (0, \infty) \iff \theta_0 = - \lim_{s \rightarrow 0} \frac{s\phi'(s)}{\phi(s)}.$$

If $\theta > 0$: asymptotic dependence

Proposition 1

If $0 < \theta_0 < \infty$, then for every $\emptyset \neq I \subset \{1, \dots, d\}$, every $(x_i)_{i \in I} \in (0, \infty)^{|I|}$ and every $(y_1, \dots, y_d) \in (0, \infty)^d$,

$$\begin{aligned} & \lim_{s \downarrow 0} \Pr[\forall i = 1, \dots, d : U_i \leq sy_i \mid \forall i \in I : U_i \leq sx_i] \\ &= \left(\frac{\sum_{i \in I^c} y_i^{-\theta_0} + \sum_{i \in I} (x_i \wedge y_i)^{-\theta_0}}{\sum_{i \in I} x_i^{-\theta_0}} \right)^{-1/\theta_0} \end{aligned}$$

This is Clayton's copula.

Lower tails of Archimedean copulas

Study regular variation property of ϕ at 0,

$$\lim_{s \rightarrow 0} \frac{\phi(st)}{\phi(s)} = t^{-\theta_0}, t \in (0, \infty) \iff \theta_0 = - \lim_{s \rightarrow 0} \frac{s\phi'(s)}{\phi(s)}.$$

If $\theta = 0$: asymptotic **in**dependence (dependence in independence)

Proposition 2

If $\theta_0 = 0$ and $\phi(0) = \infty$, for every $\emptyset \neq I \subset \{1, \dots, d\}$, every $(x_i)_{i \in I} \in (0, \infty)^{|I|}$ and every $(y_1, \dots, y_d) \in (0, \infty)^d$,

$$\begin{aligned} & \lim_{s \downarrow 0} \Pr[\forall i \in I : U_i \leq sy_i; \forall i \in I^c : U_i \leq \chi_s(y_i) \mid \forall i \in I : U_i \leq sx_i] \\ &= \prod_{i \in I} \left(\frac{y_j}{x_j} \wedge 1 \right)^{|I|^{-\kappa}} \prod_{i \in I^c} \exp(-|I|^{-\kappa} y_i^{-1}), \end{aligned}$$

Upper tails of Archimedean copulas

Study regular variation property of ϕ at 1,

$$\lim_{s \rightarrow 0} \frac{\phi(1 - st)}{\phi(1 - s)} = t^{\theta_1}, t \in (1, \infty) \iff \theta_1 = - \lim_{s \rightarrow 0} \frac{s\phi'(1 - s)}{\phi(1 - s)}.$$

If $\theta > 1$: asymptotic dependence

Proposition 3

If $1 < \theta_0 < \infty$, then for every $\emptyset \neq I \subset \{1, \dots, d\}$, every $(x_i)_{i \in I} \in (0, \infty)^{|I|}$ and every $(y_1, \dots, y_d) \in (0, \infty)^d$,

$$\lim_{s \downarrow 0} \Pr[\forall i = 1, \dots, d : U_i \geq 1 - sy_i \mid \forall i \in I : U_i \geq 1 - sx_i] = \frac{r_d(z_1, \dots, z_d; \theta_1)}{r_{|I|}((x_i)_{i \in I}; \theta_1)}$$

where $z_i = x_i \wedge y_i$ for $i \in I$ and $z_i = y_i$ for $i \in I^c$ and

$$r_k(u_1, \dots, u_k; \theta_1) = \sum_{\emptyset \neq J \subset \{1, \dots, k\}} (-1)^{|J|-1} \left(\prod_{i \in J} u_i^{\theta_1} \right)^{1/\theta_1}$$

for integer $k \geq 1$ and $(u_1, \dots, u_k) \in (0, \infty)^k$.

Upper tails of Archimedean copulas

Study regular variation property of ϕ at 1,

$$\lim_{s \rightarrow 0} \frac{\phi(1 - st)}{\phi(1 - s)} = t^{\theta_1}, t \in (1, \infty) \iff \theta_1 = - \lim_{s \rightarrow 0} \frac{s\phi'(1 - s)}{\phi(1 - s)}.$$

If $\theta > 1$ and $\phi'(1) < 0$: asymptotic independence, or near independence

Proposition 4

If $1 < \theta_1 = 1$ and $\phi'(1) < 0$, then for all $(x_i)_{i \in I} \in (0, \infty)^{|I|}$ and $(y_1, \dots, y_d) \in (0, 1]^d$,

$$\begin{aligned} & \lim_{s \downarrow 0} \Pr[\forall i \in I : U_i \geq 1 - sx_i y_i; \forall i \in I^c : U_i \leq y_i \mid \forall i \in I : U_i \geq 1 - sx_i] \\ &= \prod_{i \in I} y_j \cdot \frac{(-D)^{|I|} \phi^{-1}(\sum_{i \in I^c} \phi(y_i))}{(-D)^{|I|} \phi^{-1}(0)}. \end{aligned}$$

Upper tails of Archimedean copulas

If $\theta > 1$ and $\phi'(1) = 0$: asymptotic independence, dependence in independence

Proposition 5

If $1 < \theta_1 = 1$ and $\phi'(1) = 0$, if $I \subset \{1, \dots, d\}$ and $|I| \geq 2$, then for every $(x_i)_{i \in I} \in (0, \infty)^{|I|}$ and every $(y_1, \dots, y_d) \in (0, \infty)^d$,

$$\lim_{s \downarrow 0} \Pr[\forall i = 1, \dots, d : U_i \geq 1 - sy_i \mid \forall i \in I : U_i \geq 1 - sx_i] = \frac{r_d(z_1, \dots, z_d)}{r_{|I|}((x_i)_{i \in I})}$$

where $z_i = x_i \wedge y_i$ for $i \in I$ and $z_i = y_i$ for $i \in I^c$ and

$$\begin{aligned} r_k(u_1, \dots, u_k) &:= \sum_{\emptyset \neq J \subset \{1, \dots, k\}} (-1)^{|J|} ({}_J u_j) \log({}_J u_j) \\ &= (k-2)! \int_0^{u_1} \cdots \int_0^{u_k} (t_1 + \cdots + t_k)^{-(k-1)} dt_1 \cdots dt_k \end{aligned}$$

for integer $k \geq 2$ and $(u_1, \dots, u_k) \in (0, \infty)^k$.

Tails of Archimedean copulas

- **upper tail**: calculate $\phi'(1)$ and $\theta_1 = -\lim_{s \rightarrow 0} \frac{s\phi'(1-s)}{\phi(1-s)}$,
 - $\phi'(1) < 0$: asymptotic independence
 - $\phi'(1) = 0$ et $\theta_1 = 1$: dependence in independence
 - $\phi'(1) = 0$ et $\theta_1 > 1$: asymptotic dependence
- **lower tail**: calculate $\phi(0)$ and $\theta_0 = -\lim_{s \rightarrow 0} \frac{s\phi'(s)}{\phi(s)}$,
 - $\phi(0) < \infty$: asymptotic independence
 - $\phi(0) = \infty$ et $\theta_0 = 0$: dependence in independence
 - $\phi(0) = \infty$ et $\theta_0 > 0$: asymptotic dependence

What do we have in dimension 2 ?

C is an Archimedean copula if $C = C_\phi$

$$C_\phi(u, v) = \phi^{-1} [\phi(u) + \phi(v)]$$

C is an extreme value copula if $C = C_A = C_\ell$

$$\begin{cases} C_A(u, v) = \exp \left(\log[uv] A \left(\frac{\log[v]}{\log[uv]} \right) \right) \\ C_\ell(u, v) = \exp[-\ell(-\log u, -\log v)] \end{cases}$$

where $A : [0, 1] \rightarrow [1/2, 1]$ is Pickands dependence function, convex, with

$$\max\{\omega, 1 - \omega\} \leq A(\omega) \leq 1, \forall \omega \in [0, 1].$$

Exemple4

$A(\omega) = 1$ yields the independent copula, C^\perp .

What do we have in dimension 2 ?

Exemple5

$\phi(t) = [-\log(t)]^\theta$ yields Gumbel copula C_θ .

$A(\omega) = [\omega^\theta + (1 - \omega)^\theta]^{1/\theta}$ yields Gumbel copula C_θ .

Definition 16

C is an Archimax copula (from CAPÉERÀ, FOUGÈRES & GENEST (JMVA, 2000)) if

$$C = C_{\phi,A}$$

$$C_{\phi,A}(u, v) = \phi^{-1} \left[[\phi(u) + \phi(v)] A \left(\frac{\phi(u)}{\phi(u) + \phi(v)} \right) \right]$$

Note that there is a frailty type construction, see C. (K, 2006): given Θ , \mathbf{X} has (survival) copula C_A , Θ has Laplace transform ϕ^{-1} .

Note that $C_{\phi,A}$ is the distorted version of copula C_A .

What do we have in dimension $d \geq 3$?

Definition 17

C is an Archimax copula (from C., FOUGÈRES, GENEST & NEŠLEHOVÁ (JMVA, 2014)) if $C = C_{\phi, \ell}$

$$C_{\phi, \ell}(u_1, \dots, u_d) = \phi^{-1} [\ell(\phi(u_1) + \dots + \phi(u_d))]$$

This function *is* a copula function.

Stochastic representation of Archimax copulas

Theorem 3

$C_{\phi,\ell}$ is the survival copula of $\mathbf{X} = \mathbf{T}/\Theta$ where Θ has Laplace transform ϕ^{-1} , independent of random vector \mathbf{T} satisfying

$$\mathbb{P}(\mathbf{T} > \mathbf{t}) = \exp[-\ell(\mathbf{t})] = C_{\ell}(e^{-\mathbf{t}}).$$

(see also LI (JMVA, 2009) and MARSHALL & OLKIN (JASA, 1988)).

Limiting behavior of Archimax copulas

One can wonder what would be the max-domain of attraction of that copula ?

$$C_{\phi, \ell} \in \text{MDA}(C_{\ell^*})$$

If $\psi = \phi^{-1}$ is such that $\psi(1 - s)$ is regularly varying at 0 with index $\theta \in [1, +\infty]$, then $C_{\phi, \ell}$ belongs to the max domain of attraction of

$$C_{\ell^*}(u_1, \dots, u_d) = \exp \left[-\ell^{\frac{1}{\theta}} (|\log(u_1)|^\theta, \dots, |\log(u_d)|^\theta) \right]$$

(see also C. & SEGERS (JMVA, 2009) and LARSSON & NEŠLEHOVÁ (AAP, 2011) in the case of Archimedean copulas).