

Solutions Exercices ARMA

Rapels:

①. Le modèle MA(q)

$$Y_t = \theta_0 + \varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2} - \dots - \theta_q \varepsilon_{t-q}$$

où ε_t sont iid $N(0, \sigma_\varepsilon^2) \forall t$.

$$\Rightarrow \text{MA}(1): Y_t = \theta_0 + \varepsilon_t - \theta_1 \varepsilon_{t-1}$$

$$\Rightarrow \text{MA}(2): Y_t = \theta_0 + \varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2}$$

| Y_t | MA(1) | MA(2) |
|---------------------------------------|---|--|
| $E[Y_t]$ | θ_0 | θ_0 |
| $\text{Var}[Y_t] = \gamma_0$ | $\sigma_\varepsilon^2 (1 + \theta_1^2)$ | $\sigma_\varepsilon^2 (1 + \theta_1^2 + \theta_2^2)$ |
| $\text{Cov}(Y_t, Y_{t-k}) = \gamma_k$ | $\begin{cases} 0 & \text{si } k > 1 \\ -\theta_1 \sigma_\varepsilon^2 & \text{si } k = 1 \end{cases}$ | $\begin{cases} 0 & \text{si } k > 2 \\ -\theta_2 \sigma_\varepsilon^2 & \text{si } k = 2 \\ (-\theta_1 + \theta_1 \theta_2) \sigma_\varepsilon^2 & \text{si } k = 1 \end{cases}$ |
| $\rho_k = \text{Corr}(Y_t, Y_{t-k})$ | $\begin{cases} 1 & \text{si } k = 0 \\ \frac{-\theta_1}{1 + \theta_1^2} & \text{si } k = 1 \\ 0 & \text{si } k > 1 \end{cases}$ | $\begin{cases} 1 & \text{si } k = 0 \\ \frac{-\theta_1 + \theta_1 \theta_2}{1 + \theta_1^2 + \theta_2^2} & \text{si } k = 1 \\ \frac{-\theta_2}{1 + \theta_1^2 + \theta_2^2} & \text{si } k = 2 \\ 0 & \text{si } k > 2 \end{cases}$ |

(2) - Modelle AR(p)

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \dots + \phi_p Y_{t-p} + \phi_0 + \varepsilon_t$$

$$\varepsilon_t \sim iid N(0, \sigma_\varepsilon^2) \quad \forall t$$

\Rightarrow AR(1): $Y_t = \phi_1 Y_{t-1} + \phi_0 + \varepsilon_t$

\Rightarrow AR(2): $Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \phi_0 + \varepsilon_t$

| Y_t | AR(1) | AR(2) |
|---|---|--|
| Stationarity | $ \phi_1 < 1$ | $ \phi_1 + \phi_2 < 1$ or $ \phi_2 < 1$ |
| $E[Y_t]$ | $\frac{\phi_0}{1 - \phi_1}$ | $\frac{\phi_0}{1 - \phi_1 - \phi_2}$ |
| $Var[Y_t] = \gamma_0$ | $\frac{\sigma_\varepsilon^2}{1 - \phi_1^2}$ | $\frac{(1 - \phi_2^2) \sigma_\varepsilon^2}{(1 + \phi_2)[(1 - \phi_2)^2 - \phi_1^2]}$ |
| $\gamma_k = Cov(Y_t, Y_{t-k})$ | $\phi_1^k \gamma_0 = \phi_1^k \gamma_{k-1}$ | $\begin{cases} \phi_1 \gamma_0 & \text{if } k=1 \\ \frac{\phi_1^2}{1 - \phi_2} \gamma_0 & \text{if } k=2 \\ \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2} & \text{if } k \geq 2 \end{cases}$ |
| $\rho_k = Corr(Y_t, Y_{t-k})$ | $\rho_k^k = \phi_1^k \rho_{k-1}$ | $\begin{cases} 1 & \text{if } k=0 \\ \frac{\phi_1}{1 - \phi_2} & \text{if } k=1 \\ \frac{\phi_1^2}{1 - \phi_2} + \phi_2 & \text{if } k=2 \\ \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} & \text{if } k \geq 2 \end{cases}$ |
| $\rho_k = Corr(Y_t, Y_{t-k})$ | $\begin{cases} \rho_k^k = \phi_1^k \rho_{k-1} \\ \text{Equat. = Yule Walker} \end{cases}$ | $\begin{cases} \bar{\rho}_1 = \frac{(1 - \bar{\rho}_2) \bar{\rho}_1}{1 - \bar{\rho}_1^2} \\ \bar{\rho}_2 = \frac{\bar{\rho}_2 - \bar{\rho}_1^2}{1 - \bar{\rho}_2^2} \\ \text{Yule Walker} \end{cases}$ |

(9). Modèle ARMA(p, q)

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \dots + \phi_p Y_{t-p} + \phi_0 + \varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2} - \dots - \theta_q \varepsilon_{t-q}$$

ARMA(1, 1) : $Y_t = \phi_1 Y_{t-1} + \phi_0 + \varepsilon_t - \theta_1 \varepsilon_{t-1}$

Conditione stationnarite:

$E[Y_t]$

$Var[Y_t]$

$Cov[Y_t, Y_{t-k}] = \gamma_k$

$corr(Y_t, Y_{t-k}) = \rho_k$

$|\phi_1| < 1$

$\frac{\phi_0}{1 - \phi_1}$

$\frac{(1 + \theta_1^2 - 2\phi_1\theta_1)\sigma_\varepsilon^2}{1 - \phi_1^2}$

$\left\{ \begin{array}{l} \frac{(\phi_1 - \theta_1)(1 - \phi_1\theta_1)\sigma_\varepsilon^2}{1 - \phi_1^2} \quad \text{si } k=1 \\ \phi_1 \gamma_{k-1} \quad \text{si } k \geq 2 \end{array} \right.$

$\left\{ \begin{array}{l} 1 \quad \text{si } k=0 \\ \frac{(\phi_1 - \theta_1)(1 - \phi_1\theta_1)}{(1 + \theta_1^2 - 2\theta_1\phi_1)} \quad \text{si } k=1 \\ \phi_1 \rho_{k-1} \quad \text{si } k \geq 2 \end{array} \right.$

(4) Prévissions:

⇒ Modèle MA (q)

a). $\varepsilon_0 = \varepsilon_{-1} = \dots = 0$

b). $\hat{\varepsilon}_t = y_t - \theta_0 + \theta_1 \varepsilon_{t-1} + \dots + \theta_q \varepsilon_{t-q}$

c). $\hat{y}_{T+h} = \mathbb{E} [Y_{T+h} | y_1, \dots, y_T]$

⇒ Modèle AR (p)

a). $y_0 = y_{-1} = \dots = \frac{\phi_0}{1 - \phi_1 - \phi_2 - \dots - \phi_p}$

b). estimation par Moindres Carrés des paramètres.

c). $\hat{y}_{T+h} = \mathbb{E} [Y_{T+h} | y_1, \dots, y_T]$

⇒ ARMA (p, q)

a). $y_0 = y_{-1} = \dots = \frac{\phi_0}{1 - \phi_1 - \phi_2 - \dots - \phi_p} = \theta_0$

b). $\varepsilon_0 = \varepsilon_{-1} = \dots = 0$

c). $\hat{\varepsilon}_t = y_t - \phi_1 y_{t-1} - \dots - \phi_p y_{t-p} - \phi_0 + \theta_1 \varepsilon_{t-1} + \dots + \theta_q \varepsilon_{t-q}$
 $t = 1, \dots, T$

d). $\hat{y}_{T+h} = \mathbb{E} [Y_{T+h} | y_1, \dots, y_T]$

A1 - Processus MA

Par définition, ~~un processus~~ si $X_t \sim MA(1)$ on écrit:

$$X_t = \varepsilon_t - \theta_1 \varepsilon_{t-1}$$

⇒ Fonction d'autocovariance : $\gamma_X(k)$ est:

$$\begin{aligned} \gamma_X(k) &= \text{Cov}(X_t, X_{t+k}) = \text{Cov}(\varepsilon_t - \theta_1 \varepsilon_{t-1}, \varepsilon_{t+k} - \theta_1 \varepsilon_{t+k-1}) \\ &= \mathbb{E}[\varepsilon_t \cdot \varepsilon_{t+k} - \theta_1 \varepsilon_t \varepsilon_{t+k-1} - \theta_1 \varepsilon_{t-1} \varepsilon_{t+k} + \theta_1^2 \varepsilon_{t-1} \varepsilon_{t+k-1}] \\ &= \mathbb{E}[\varepsilon_t \varepsilon_{t+k}] - \theta_1 \mathbb{E}[\varepsilon_t \varepsilon_{t+k-1}] - \theta_1 \mathbb{E}[\varepsilon_{t-1} \varepsilon_{t+k}] \\ &\quad + \theta_1^2 \mathbb{E}[\varepsilon_{t-1} \varepsilon_{t+k-1}] \end{aligned}$$

$$\Rightarrow \gamma_X(k) = \begin{cases} \sigma_\varepsilon^2 + \theta_1^2 \sigma_\varepsilon^2 = (1 + \theta_1^2) \sigma_\varepsilon^2 & \text{si } k=0 \\ -\theta_1 \sigma_\varepsilon^2 & \text{si } |k|=1 \\ 0 & \text{si } |k| > 1 \end{cases}$$

⇒ Fonction d'autocorrélation est donc:

$$\rho_X(k) = \frac{\gamma_X(k)}{\gamma_X(0)} = \begin{cases} 1 & \text{si } k=0 \\ -\frac{\theta_1 \sigma_\varepsilon^2}{\sigma_\varepsilon^2 (1 + \theta_1^2)} = -\frac{\theta_1}{1 + \theta_1^2} & \text{si } |k|=1 \\ 0 & \text{si } |k| > 1 \end{cases}$$

MA - un processus stationnaire, par définition

En faisant varier θ_1 sur $]-1, 1[$ on a :

$$k=1, \theta_1=0 \Rightarrow \rho_X(1) = 0$$

$$\theta_1 = \frac{1}{2} \Rightarrow \rho_X(1) = \frac{-\frac{1}{2}}{1 + \frac{1}{4}} = -0,4$$

$$\theta_1 = -\frac{1}{2} \Rightarrow \rho_X(1) = \frac{\frac{1}{2}}{1 + \frac{1}{4}} = 0,4$$

$$\theta_1 = 1 \Rightarrow \rho_X(1) = -\frac{1}{2}$$

$$\theta_1 = -1 \Rightarrow \rho_X(1) = \frac{1}{2}$$

La corrélation maximale est donc $\frac{1}{2}$ ~~et~~ 0.

A2

On utilise un MA(1), soit $X_t \sim \text{MA}(1)$

On dispose de la valeur de ~~la~~ l'autocorrélation d'ordre 1 de la série X_t :

$$\rho_X(1) = 0,4$$

\Rightarrow Calculons θ_1

Par définition, un MA(1) a écrit :

$$X_t = \varepsilon_t - \theta_1 \varepsilon_{t-1}$$

$$\Rightarrow \rho_X(1) = \frac{-\theta_1}{1 + \theta_1^2} = 0,4$$

$$\text{il vient } -\theta_1 = 0,4 + 0,4 \theta_1^2 \quad \frac{-2,5 + 1,5}{2} = -0,5$$

\Rightarrow Equation du second degré

$$\text{solutions: } \theta_1 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \begin{cases} \frac{-2,5 + 1,5}{2} = -0,5 \\ \frac{-2,5 - 1,5}{2} = -2 \end{cases}$$

(6)

⇒ la réponse est $\theta_1 = -0.5$, puisque:

$$\text{Var}(X_t) = \sigma_\varepsilon^2 (1 + \theta_1^2) \quad (\text{MA}(1))$$

Pour $\theta_1 = 0.5$

$$\text{on a } \text{Var}(X_t) = \sigma_\varepsilon^2 \cdot 1.25$$

Pour $\theta_1 = -2$,

$$\text{Var}(X_t) = \sigma_\varepsilon^2 (1 + 4) = 5\sigma_\varepsilon^2.$$

Avec $\sigma_\varepsilon^2 = \text{fixe} = \text{constante}$,

$$\begin{array}{l} \text{Var}(X_t) < \text{Var}(X_t) \\ \text{avec } \theta_1 = -0.5 \quad \text{avec } \theta_1 = -2. \end{array}$$

On choisit la valeur de θ_1 qui donne une variance minimale.

A3 - Modèle MA(1)

$$Y_t = \mu + \varepsilon_t - \theta_1 \varepsilon_{t-1}, \quad t = 1, \dots, N$$

où $\mu = \theta_0$ (de la définition)

On dispose: $\mu = 1$, $\theta_1 = -0.5$ et $Y_N = 8$

⇒ Déterminons ${}_N \hat{Y}_{N+1}$: prévision à l'horizon $h=1$

$$\begin{aligned} {}_N \hat{Y}_{N+1} &= \hat{Y}_{N(1)} = \mathbb{E} [Y_{N+1} | Y_1, \dots, Y_N] \\ &= \mathbb{E} [\mu + \varepsilon_{N+1} - \theta_1 \varepsilon_N | Y_1, \dots, Y_N] \\ &= \mathbb{E} [\mu | Y_1, \dots, Y_N] + \mathbb{E} [\varepsilon_{N+1} | Y_1, \dots, Y_N] \\ &\quad - \theta_1 \mathbb{E} [\varepsilon_N | Y_1, \dots, Y_N] \end{aligned}$$

(7)

$$\Rightarrow \hat{Y}_{N+1} = \hat{Y}_{N+1} = \mathbb{E}[Y_{N+1} | Y_1, \dots, Y_N]$$

$$= \mu + 0 - \theta_1 \mathbb{E}[\varepsilon_N | Y_1, \dots, Y_N]$$

A7

$$\text{MA(1)}: Y_t = \mu + \varepsilon_t - \theta_1 \varepsilon_{t-1}$$

$$\text{- Calculons } \mathbb{E}[(Y_t - \mu)(Y_{t-1} - \mu)]$$

$$\mathbb{E}[(Y_t - \mu)(Y_{t-1} - \mu)] = \mathbb{E}[\varepsilon_t - \varepsilon_{t-1}] - \theta_1 \mathbb{E}[\varepsilon_t \varepsilon_{t-2}]$$

$$- \theta_1 \mathbb{E}[\varepsilon_{t-1}^2] + \theta_1^2 \mathbb{E}[\varepsilon_{t-1} \varepsilon_{t-2}]$$

$$\gamma_Y(1) = -\theta_1 \sigma_\varepsilon^2 \neq -\theta_1$$

A8

$$\text{MA(1)}, Y_t = \mu + \varepsilon_t - \phi_1 \varepsilon_{t-1}$$

$$\rho_Y(k) \stackrel{?}{=} 0, \forall |k| > 1 \quad ?$$

- est vrai. Pourquoi:

$$\rho_Y(k) = \begin{cases} 1 & \text{si } k=0 \\ -\frac{\phi_1}{1+\phi_1^2} & \text{si } k=1 \\ 0 & \text{si } k \geq 1 \end{cases}$$

- par définition -

$$\boxed{A9} - \text{MA}(2), Y_t = \mu + \varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2}$$

$$\text{avec } \mu = 0, \theta_1 = 0.2, \theta_2 = 0.7$$

Calculons $\rho_Y(2)$

$$\gamma_Y(2) = \text{Cov}(Y_t, Y_{t+2}) = \mathbb{E} \left[(\varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2}) \right. \\ \left. \times (\varepsilon_{t+2} - \theta_1 \varepsilon_{t+1} - \theta_2 \varepsilon_{t+2-2}) \right]$$

$$= \mathbb{E} \left[\varepsilon_t \cdot \varepsilon_{t+2} - \theta_1 \varepsilon_t \cdot \varepsilon_{t+1} - \theta_2 \varepsilon_t \cdot \varepsilon_{t+2-2} \right. \\ \left. - \theta_1 \varepsilon_{t-1} \cdot \varepsilon_{t+2} + \theta_1^2 \varepsilon_{t-1} \cdot \varepsilon_{t+1} \right. \\ \left. + \theta_1 \theta_2 \varepsilon_{t-1} \cdot \varepsilon_{t+2-2} - \theta_2 \varepsilon_{t-2} \cdot \varepsilon_{t+2} \right. \\ \left. + \theta_1 \theta_2 \varepsilon_{t-2} \cdot \varepsilon_{t+1} + \theta_2^2 \varepsilon_{t-2} \cdot \varepsilon_{t+2-2} \right]$$

$$= \begin{cases} \sigma_\varepsilon^2 (1 + \theta_1^2 + \theta_2^2) & k=0 \\ -\theta_2 \sigma_\varepsilon^2 & k=2 \end{cases}$$

$$\Rightarrow \rho_Y(2) = \frac{\gamma_Y(2)}{\gamma_Y(0)} = \frac{-\theta_2}{1 + \theta_1^2 + \theta_2^2}$$

$$\text{Avec: } \rho_Y(2) = \frac{-0.7}{1 + 0.2^2 + 0.7^2} = \frac{-0.7}{1 + 0.04 + 0.49} \\ = \frac{-0.7}{1.53} = -0.457 \approx -0.46$$

\Rightarrow Réponse A

A10 | $T=50$ observations

MA(1) : sans constante

$$Y_t = \varepsilon_t - \theta_1 \varepsilon_{t-1}$$

• le consultant croit que $\theta > 0.9$.
Vérification:

On sait que pour un MA(1), la fonction d'autocorrélation d'ordre 1 est :

$$\rho_Y(1) = -\frac{\theta_1}{1+\theta_1^2} = \frac{\sum (Y_t - \bar{Y})(Y_{t+1} - \bar{Y})}{\sum (Y_t - \bar{Y})^2} = 0.93$$

Si $\theta_1 > 0.9$, alors :

$$-\frac{\theta_1}{1+\theta_1^2} > 0.93$$

~~On a~~ ~~une~~

On a : $-\theta_1 = 0.93 + 0.93 \theta_1^2$

$$0.93 \theta_1^2 + \theta_1 - 0.93 = 0$$

$$\Delta = b^2 - 4ac = 1 + 4 \times 0.93 \times 0.93 = 4.4596$$

$$\theta_1 = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-1 - \sqrt{4.4596}}{2 \times 0.93} \ll 0.9$$

$$\frac{-1 + \sqrt{4.4596}}{2 \times 0.93}$$

⇒ Réponse fausse.