

Multi-Attribute Utility & Copulas (based on Ali E. Abbas contributions)

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<http://freakonometrics.hypotheses.org>

Olivier's Talk, part 2, on Independence & Additivity

Some fundamental concepts of multiattribute (expected) utility theory

- ▶ Utility independence
- ▶ Mutual utility independence
- ▶ Conditional utility functions
- ▶ Additive independence
- ▶ Scaling constants
- ▶ Additive/Multiplicative representation

Olivier's Talk, part 2, on Utility Independence

Utility Independence

- ▶ Utility function $u(y, z)$
- ▶ Y is utility independent of Z when conditional preferences for lotteries on Y given z do not depend on the particular level of z
- ▶ Example: certainty equivalents does not depends on the value of z^0 .

see also [Keeney & Raiffa \(1976\)](#)

Definition. We shall say that Y is *utility independent* of Z when conditional preferences for lotteries on Y given z do not depend on the particular level of z .

Olivier's Talk, part 2, on Mutual Utility Independence

Mutual Utility Independence

- ▶ When Y utility independent of Z and Z utility independent of Y
- ▶ Y and Z are said to be mutually utility independent
- ▶ Independence under certainty is a special case of mutual utility independence for degenerate lotteries
- ▶ when mutual utility independence holds
 - ▶ choose a conditional utility function $u(., z^1)$
 - ▶ choose a conditional utility function $u(y^1, .)$
 - ▶ choose the utility of an arbitrary consequence (y^2, z^2)
 - ▶ allows to entirely specify $u(x, y)$ for all y, z .
- ▶ CAUTION: mutual utility independence does not imply an additive utility function.

Olivier's Talk, part 2, on Additive Utility Independence

Additive Independence and additive utility function

- ▶ Y and Z are additive independent if and only if the two-attribute utility function is additive
- ▶ $u(y, z) = u(y, z^0) + u(y^0, z)$
- ▶ or $u(y, z) = k_Y u_Y(y) + k_Z u_Z(z)$ with
 - ▶ $u(y, z)$ normalized between $u(y^0, z^0) = 0$ and $u(y^1, z^1) = 1$
 - ▶ $u_Y(y)$ is a conditional utility function normalized between $u_Y(y^0) = 0$ and $u_Y(y^1) = 1$
 - ▶ $u_Z(z)$ is a conditional utility function normalized between $u_Z(z^0) = 0$ and $u_Z(z^1) = 1$
 - ▶ $k_Y = u(y^1, z^0)$ and $k_Z = u(y^0, z^1)$.

Olivier's Talk, part 2, on Additive Utility Independence

Theorem 5.1. *Attributes Y and Z are additive independent if and only if the two-attribute utility function is additive. The additive form may be written either as*

$$u(y, z) = u(y, z^o) + u(y^o, z), \quad (5.9)$$

or as

$$u(y, z) = k_Y u_Y(y) + k_Z u_Z(z) \quad (5.10)$$

where

1. $u(y, z)$ is normalized by $u(y^o, z^o) = 0$ and $u(y^1, z^1) = 1$ for arbitrary y^1 and z^1 such that $(y^1, z^o) \succ (y^o, z^o)$ and $(y^o, z^1) \succ (y^o, z^o)$.
2. $u_Y(y)$ is a conditional utility function on Y normalized by $u_Y(y^o) = 0$ and $u_Y(y^1) = 1$.
3. $u_Z(z)$ is a conditional utility function on Z normalized by $u_Z(z^o) = 0$ and $u_Z(z^1) = 1$.
4. $k_Y = u(y^1, z^o)$.
5. $k_Z = u(y^o, z^1)$.

Olivier's Talk, part 2, on Mutual Utility Independence

The Multilinear Utility Function

- ▶ When Y and Z are said to be mutually utility independent, then $u(x, y)$ can be decomposed using the multilinear form:

$$u(y, z) = k_Y u_Y(y) + k_Z u_Z(z) + k_{YZ} u_Y(y) u_Z(z) \quad (3)$$

or

$$u(y, z) = u(y, z_0) + u(y_0, z) + k u(y, z_0) u(y_0, z) \quad (4)$$

- ▶ mutual utility independence reduces the dimensionality of the problem from 2 to 2×1
- ▶ $k_{YZ} = 1 - k_Y - k_Z$
- ▶ when $k_{YZ} \neq 0$ equivalent to a multiplicative representation
- ▶ when $k_{YZ} = 0$ equivalent to an additive representation
- ▶ scaling constants do not indicate the relative importance of attributes

Olivier's Talk, part 2, on Mutual Utility Independence

Theorem 5.2. *If Y and Z are mutually utility independent, then the two-attribute utility function is multilinear. In particular, u can be written in the form*

$$u(y, z) = u(y, z_0) + u(y_0, z) + ku(y, z_0)u(y_0, z), \quad (5.16)$$

or

$$u(y, z) = k_Y u_Y(y) + k_Z u_Z(z) + k_{YZ} u_Y(y) u_Z(z), \quad (5.17)$$

where

1. $u(y, z)$ is normalized by $u(y_0, z_0) = 0$ and $u(y_1, z_1) = 1$ for arbitrary y_1 and z_1 such that $(y_1, z_0) \succ (y_0, z_0)$ and $(y_0, z_1) \succ (y_0, z_0)$.
2. $u_Y(y)$ is a conditional utility function on Y normalized by $u_Y(y_0) = 0$ and $u_Y(y_1) = 1$.
3. $u_Z(z)$ is a conditional utility function on Z normalized by $u_Z(z_0) = 0$ and $u_Z(z_1) = 1$.
4. $k_Y = u(y_1, z_0)$.
5. $k_Z = u(y_0, z_1)$.
6. $k_{YZ} = 1 - k_Y - k_Z$ and $k = k_{YZ}/k_Y k_Z$.

What are we looking for?

Ideally, we would like to obtain a representation of the utility function such that

$$u(x_1, x_2, \dots, x_n) = f[f_1(x_1), f_2(x_2), \dots, f_n(x_n)], \quad (5.5)$$

where f_i is a function of attribute X_i only, for $i = 1, 2, \dots, n$, and where f has a simple form, an additive or multiplicative form, for example.

See [Sklar \(1959\)](#) for cumulative distribution function for random vector $\mathbf{X} \in \mathbb{R}^n$,

$$F(x_1, \dots, x_n) = C[F_1(x_1), \dots, F_n(x_n)]$$

where $F(\mathbf{x}) = \mathbb{P}[\mathbf{X} \leq \mathbf{x}]$ and $F_i(x_i) = \mathbb{P}[X_i \leq x_i]$.

What are we looking for?

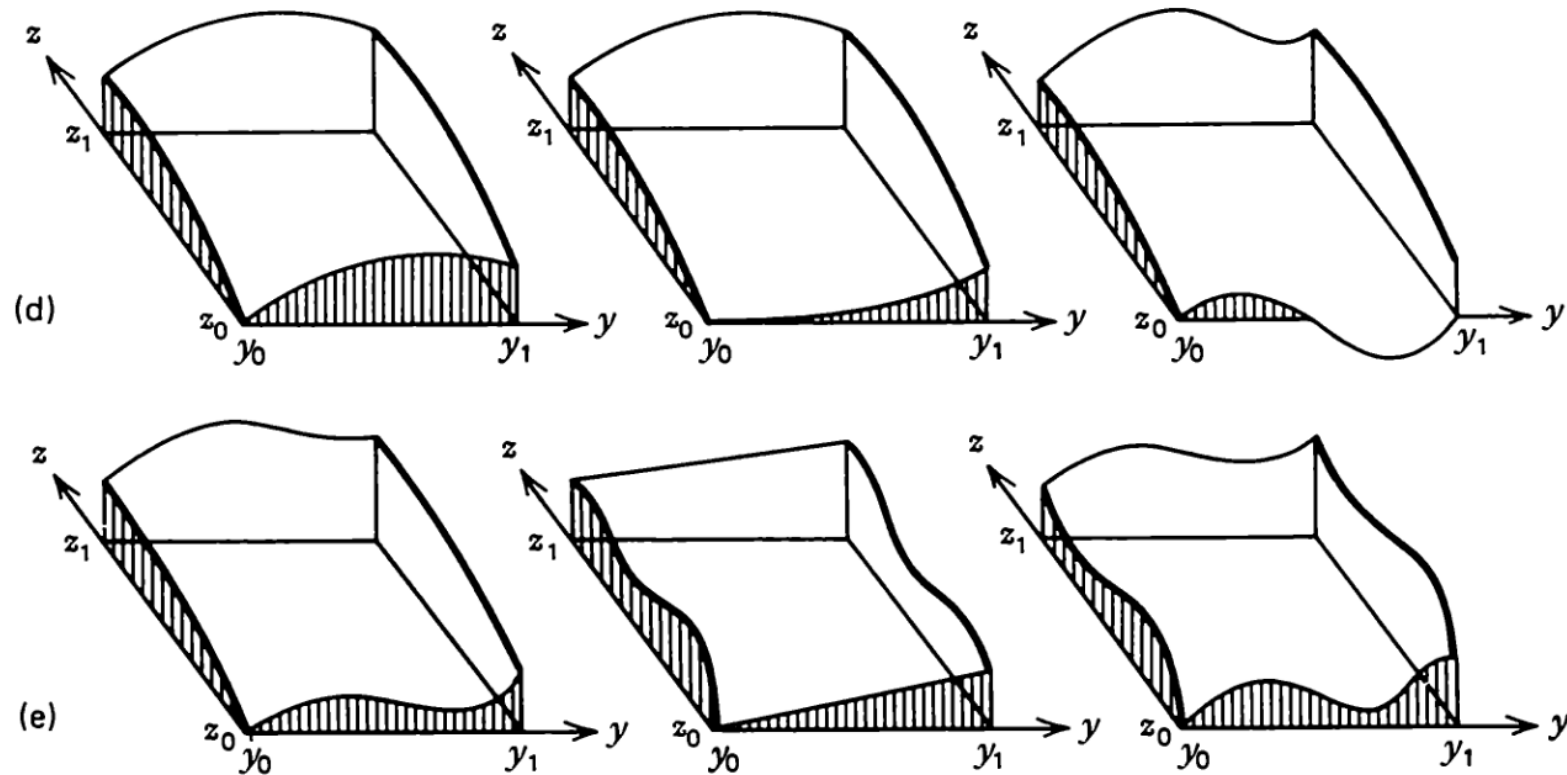


Fig. 5.13. Different shapes of utility functions $u(y, z)$ where Z is utility independent of Y .

Historical Perspective

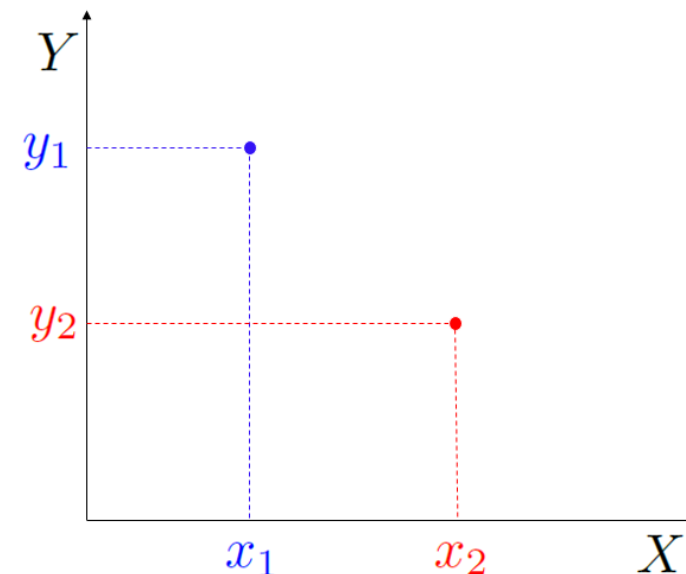
When everything else remains constant which do you prefer

(x_1, y_1) or (x_2, y_2)

X can be consumption

Y can be health

(remaining life time expectancy)

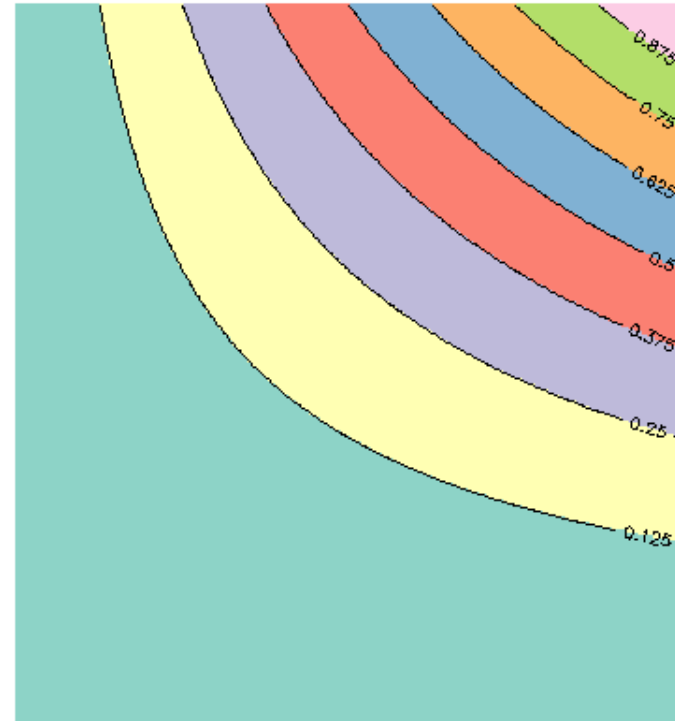
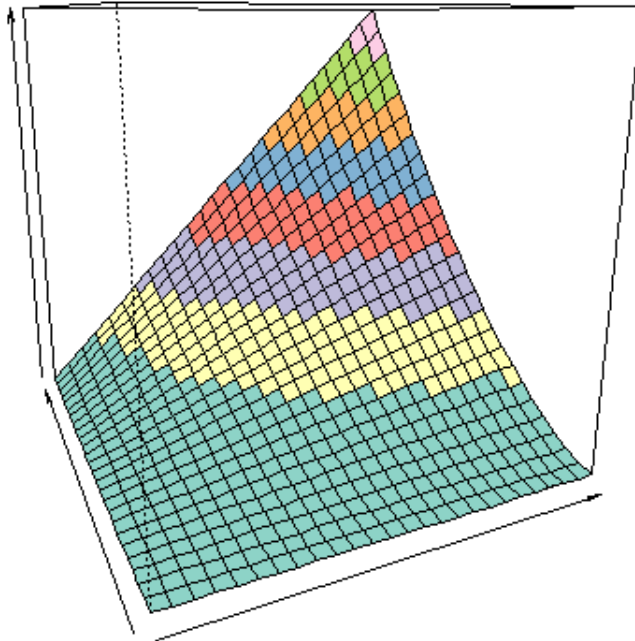


Matheson & Howard (1968) : use a deterministic real-valued function $V : \mathbb{R}^d \rightarrow \mathbb{R}$ and then use a utility function over the value function,

$$U(\mathbf{x}) = U(x_1, \dots, x_d) = u(V(x_1, \dots, x_d)),$$

e.g. $U(\mathbf{x}) = u(x_1 + \dots + x_d)$ or $u(\min\{x_1, \dots, x_d\})$.

Historical Perspective



See Matheson & Abbas (2005), e.g. $V(x, y) = xy^\eta$,

see also Sheldon's acoustic sweet spot or peanut butter/jelly sandwich preference function

Historical Perspective

Alternative approach: assess utilities over individual attributes, and combine them into a functional form

Keeney & Raiffa (1976) : use some utility independence assumption

Mutual utility independence : $U(x, y) = k_x u_x(x) + k_y u_y(y) + k_{xy} u_x(x) u_y(y)$
where $k_{xy} = 1 - k_x - k_y$

Additive and Product forms

$$U(x, y) = k_x u_x(x) + k_y u_y(y) \text{ with } k_x + k_y = 1$$

$$U(x, y) = k_{xy} u_x(x) u_y(y)$$

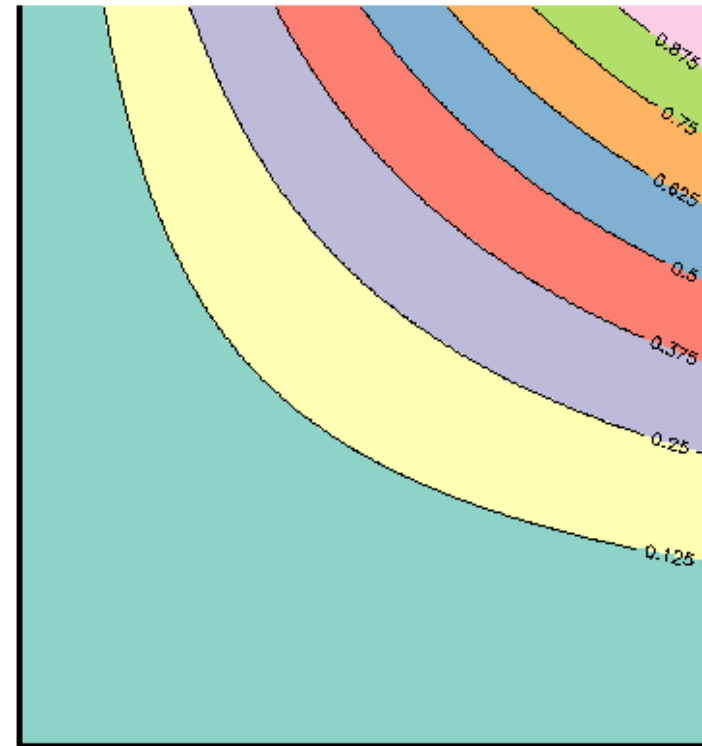
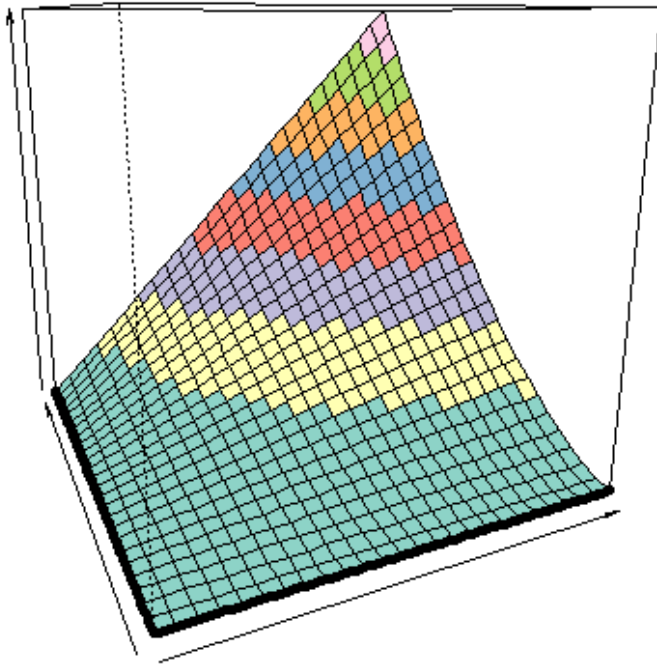
Utility Independence is an interesting property, but it might be a simplifying one.

How to Construct Multi-Attribute Utility Functions

From [Abbas & Howard \(2005\)](#), in dimension $d = 2$,

$U(x, y) \in [0, 1]$ (normalization)

$U(\underline{x}, y) = U(x, \underline{y}) = 0$ (attribute dominance condition)

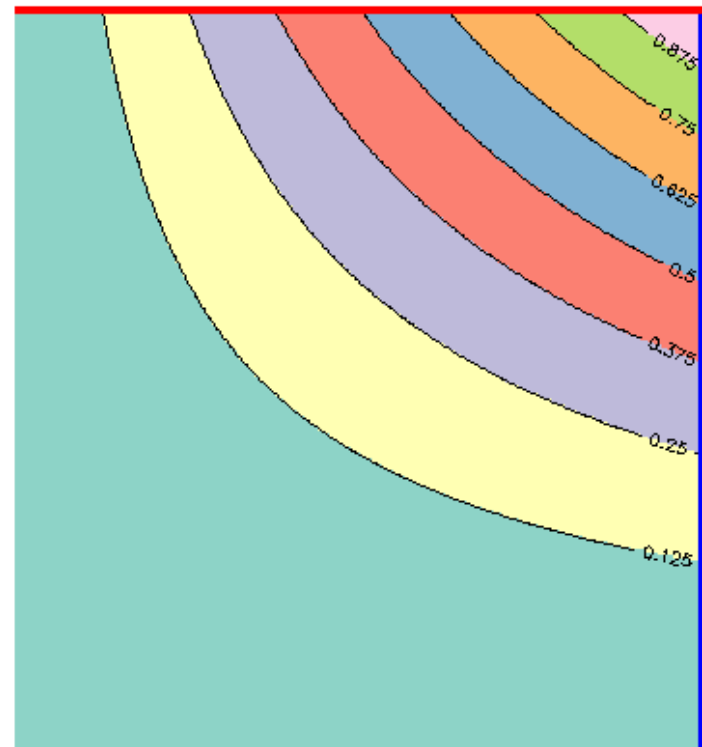
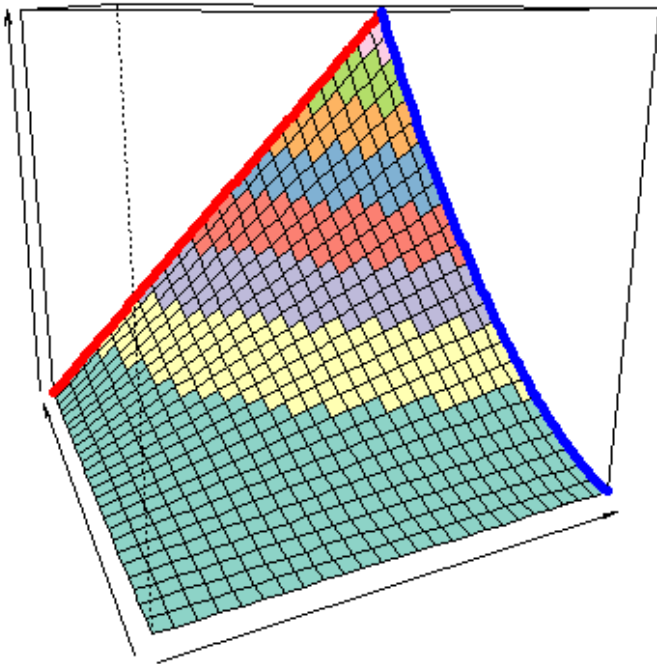


How to Construct Multi-Attribute Utility Functions

Non-decreasing with arguments:

- given y , $x_1 < x_2$ implies $(x_1, y) \preceq (x_2, y)$
- given x , $y_1 < y_2$ implies $(x, y_1) \preceq (x, y_2)$

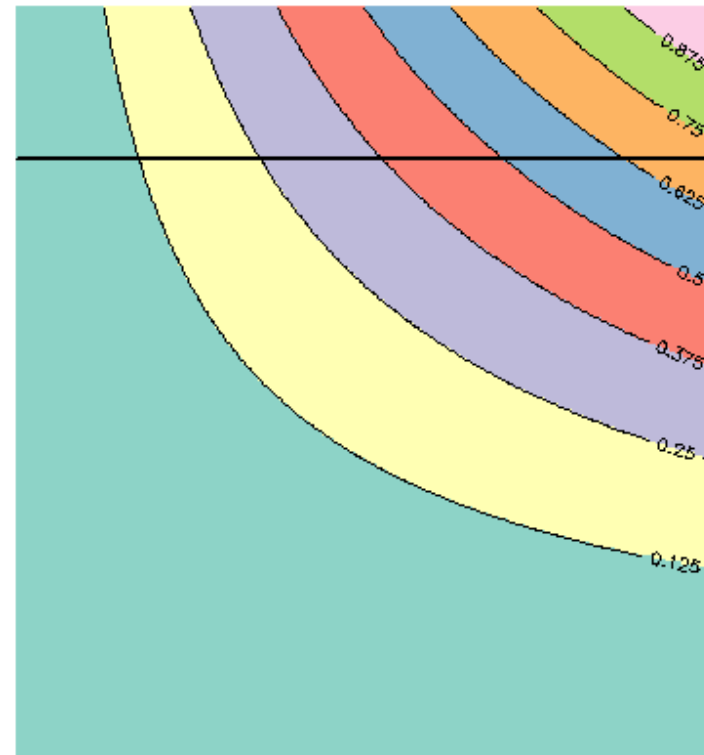
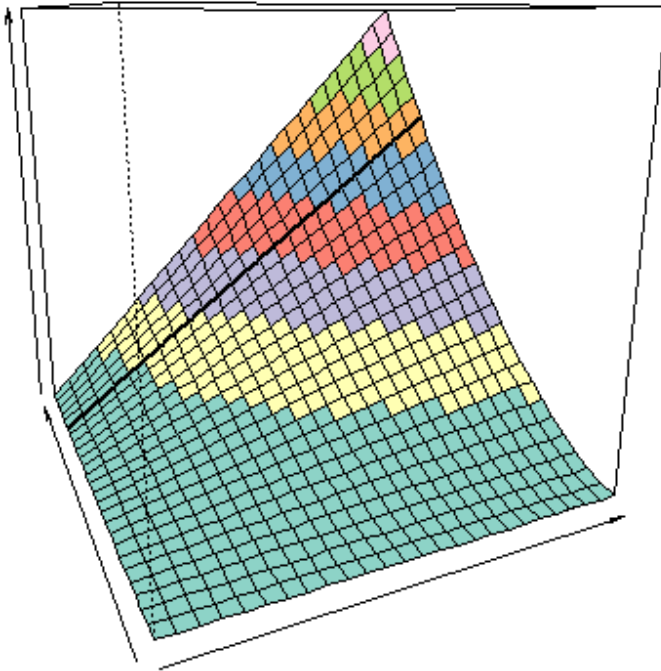
$$U(x, \bar{y}) = u_x(x) \text{ and } U(\bar{x}, y) = u_y(y)$$



Conditional Utility

We can define conditional utility

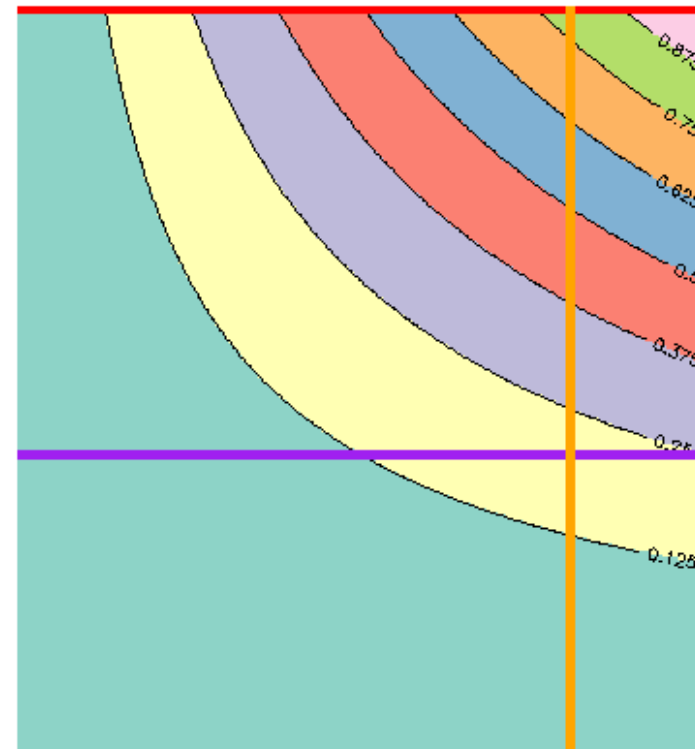
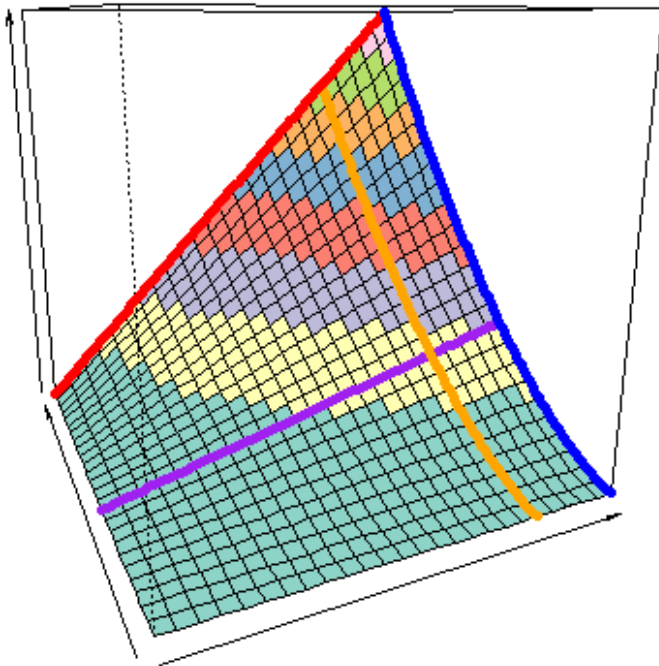
$$U_{y|x}(y|x) = \frac{U(x, y)}{u_x(x)}$$



Conditional Utility

Bayes' Rule for Attribute Dominance Utility

$$U(x, y) = u_x(x) \cdot U_{y|x}(y|x) = u_y(y) \cdot U_{x|y}(x|y).$$



Copula Structures for Attribute Dominance Utility

With two attributes, consider $U(x, y) = C(u_x(x), u_y(y))$

Since copulas are related to probability measures, function C are 2-increasing.

C is the cumulative distribution function of some U , and

$$\mathbb{P}(U \in [a, b]) \geq 0$$

implies positive mixed partial derivatives, $\frac{\partial^2 C(u, v)}{\partial u \partial v} \geq 0$ (weaker condition exist).

Not a necessary condition for attribute dominance utility theory...

Understanding the Two Attribute Framework

C might be on a normalized domain, with a normalized range $C : [0, 1]^2 \rightarrow [0, 1]$, with $C(0, 0) = 0$ and $C(1, 1) = 1$.

From Keeney & Raiffa (1976)

X independent of Y (preferences for lotteries over x do not depend on y)

$$U(x, y) = k_2(y)U(x, y_0) + d_2(y)$$

Y independent of X (preferences for lotteries over y do not depend on x)

$$U(x, y) = k_1(x)U(x_0, y) + d_1(x)$$

C should satisfy some **marginal property**: there are u_0 and v_0 such that

$$C(u_0, v) = \alpha_{u_0}v + \beta_{u_0} \text{ and } C(u, v_0) = \alpha_{v_0}u + \beta_{v_0}.$$

Margins are non decreasing, $\frac{\partial C(u, v)}{\partial u} > 0$ and $\frac{\partial C(u, v)}{\partial v} > 0$.

Understanding the Two Attribute Framework

Abbas & Howard (2005) defined some Class 1 Multiattribute Utility Copulas such that

$$C(1, v) = \alpha_{u_0} v + \beta_{u_0} \text{ and } C(u, 1) = \alpha_{v_0} u + \beta_{v_0}.$$

Proposition *Any multi-attribute utility function $U(x_1, \dots, x_n)$ that is continuous, bounded and strictly increasing in each argument can be expressed in terms of its marginal utility functions $u_1(x_1), \dots, u_n(x_n)$ and some class 1 multiattribute utility copula*

$$U(x_1, \dots, x_n) = C[u_1(x_1), \dots, u_n(x_n)].$$

Archimedean Copulas

On probability cumulative distribution functions

$$C(u_1, \dots, u_d) = \phi^{-1}(\phi(u_1) + \dots + \phi(u_d)) = \phi^{-1} \left(\sum_{j=1}^n \phi(u_j) \right)$$

with $\phi : [0, 1] \rightarrow \mathbb{R}_+$ an additive generator, or with $\psi = \phi^{-1}$ completely monotone

$$C(u_1, \dots, u_d) = \psi(\psi^{-1}(u_1) + \dots + \psi^{-1}(u_d)) = \psi \left(\sum_{j=1}^n \psi^{-1}(u_j) \right)$$

One can define some multiplicative generator, $\lambda(t) = e^{-\phi(t)}$

$$C(u_1, \dots, u_d) = \lambda^{-1}(\lambda(u_1) \times \dots \times \lambda(u_d)) = \lambda^{-1} \left(\prod_{j=1}^n \lambda^{-1}(u_j) \right)$$

E.g. $\phi(t) = -\log(t)$ or $\lambda(t) = t$, independent copula, $C = \Pi = C^\perp$

Archimedean Utility Copulas

In the context of utility functions,

$$C(v_1, \dots, v_d) = \alpha \psi^{-1} \left(\prod_{i=1}^d \psi(\gamma_i + [1 - \gamma_i]v_i) \right) + [1 - \alpha]$$

with $\gamma_i \in [0, 1]$, and such that $a = \left[\psi^{-1} \left(\prod_{i=1}^d \psi(\gamma_i) \right) \right]^{-1}$.

ψ continuous strictly increasing, $\psi(0) = 0$ and $\psi(1) = 1$.

E.g. $\psi(t) = t$, then

$$C(v_1, v_2) = \alpha[\gamma_1 + (1 - \gamma_1)v_1][\gamma_2 + (1 - \gamma_2)v_2] + (1 - \alpha)$$

i.e. **multiplicative form of mutual independence.**

Alternative to this Two Attribute Framework

By relaxing the condition of ‘attribute dominance’, [Abbas & Howard \(2005\)](#) defined some **Class 2 Multiattribute Utility Copulas** such that

$$C(0, v) = \alpha_{u_0} v + \beta_{u_0} \text{ and } C(u, 0) = \alpha_{v_0} u + \beta_{v_0}.$$

Define a multiattribute utility copula C as a multivariate function of d variables satisfying $C : [0, 1]^d \rightarrow [0, 1]$, with $C(\mathbf{0}) = 0$, $C(\mathbf{1}) = 1$, the following marginal property

$$C(0, \dots, 0, v_i, 0, \dots, 0) = \alpha_i v_i + \beta_i, \text{ with } \alpha_i > 0$$

and with $\partial C(\mathbf{v}) / \partial v_i > 0$

Alternative to this Two Attribute Framework

To define some **Class 2 Archimedean utility copulas**, let h be continuous on $[0, d]$, strictly increasing, with $h(0) = 1$ and $h(1)^d \leq h(d)$. Then set

$$C(v_1, \dots, v_d) = \frac{h^{-1} \left(\prod_{j=1}^d h(\omega_j v_j) \right)}{h^{-1} \left(\prod_{j=1}^d h(\omega_j) \right)}, \text{ with } 0 \leq \omega_j \leq 1.$$

E.g. $h(t) = e^t$, then $C(U_1(x_1), \dots, U_d(x_d)) = \tilde{\omega}_1 U_1(x_1) + \dots + \tilde{\omega}_d U_d(x_d)$, where $\tilde{\omega}_j = \omega_j / [\omega_1 + \dots + \omega_d]$, i.e. additive form of utility independence.

One-Switch Utility Independence

Introduced in [Abbas & Bell \(2011\)](#)

Consider two attributes x and y , utility function $U(x, y)$.

x is one-switch independent of y if and only if the ordering of any two lotteries over x switches at most once as y increases

Proposition x is one-switch independent of y if and only if

$$U(x, y) = g_0(y) + g_1(y)[f_1(x) + f_2(x) \cdot \varphi(y)]$$

where g_1 has a constant sign, and φ is monotone.

One-Switch Utility Independence

$$U(x, y) = g_0(y) + g_1(y)[f_1(x) + f_2(x)\varphi(y)]$$

It is possible to express those function in terms of utility

- $g_0(y) = U(\underline{x}, y)$
- $g_1(y) = [U(\bar{x}, y) - U(\underline{x}, y)]$
- $f_1(x) = U(x|\underline{y})$
- $f_2(x) = [U(x|\bar{y}) - U(x|\underline{y})]$

$$\varphi(y) = \frac{U(x|\bar{y}) - U(x|\underline{y})}{U(x|\bar{y}) - U(x|\underline{y})}$$

Utility Trees and Bidirectional Utility Diagrams

From [Abbas \(2011\)](#), let $\mathbf{x} = (x_i, \mathbf{x}_{(i)})$

Consider the normalized conditional utility for x_i at \mathbf{x} ,

$$U(x_i|\mathbf{x}_{(i)}) = \frac{U(x_i, \mathbf{x}_{(i)}) - U(\underline{x}_i, \mathbf{x}_{(i)})}{U(\bar{x}_i, \mathbf{x}_{(i)}) - U(\underline{x}_i, \mathbf{x}_{(i)})}$$

Note that

$$U(x_i, \mathbf{x}_{(i)}) = U(\bar{x}_i, \mathbf{x}_{(i)}) \cdot U(x_i|\mathbf{x}_{(i)}) + U(\underline{x}_i, \mathbf{x}_{(i)}) \cdot [1 - U(x_i|\mathbf{x}_{(i)})]$$

Thus, for two attributes

$$U(x, y) = U(\bar{x}, y) \cdot U(x|y) + U(\underline{x}, y) \cdot [1 - U(x|y)]$$

Utility Trees and Bidirectional Utility Diagrams

$$U(x, y) = U(\bar{x}, y) \cdot U(x|y) + U(\underline{x}, y) \cdot [1 - U(x|y)]$$

But it is also possible to expand it

$$U(x, y) = \underbrace{U(\bar{x}, y)}_{\substack{=U(y|\bar{x}) \cdot U(\bar{x}, \bar{y}) \\ + [1 - U(y|\bar{x})] \cdot U(\bar{x}, \underline{y})}} U(x|y) + \underbrace{U(\underline{x}, y)}_{\substack{=U(y|\underline{x}) \cdot U(\underline{x}, \bar{y}) \\ + [1 - U(y|\underline{x})] \cdot U(\underline{x}, \underline{y})}} [1 - U(x|y)]$$

which give four terms.

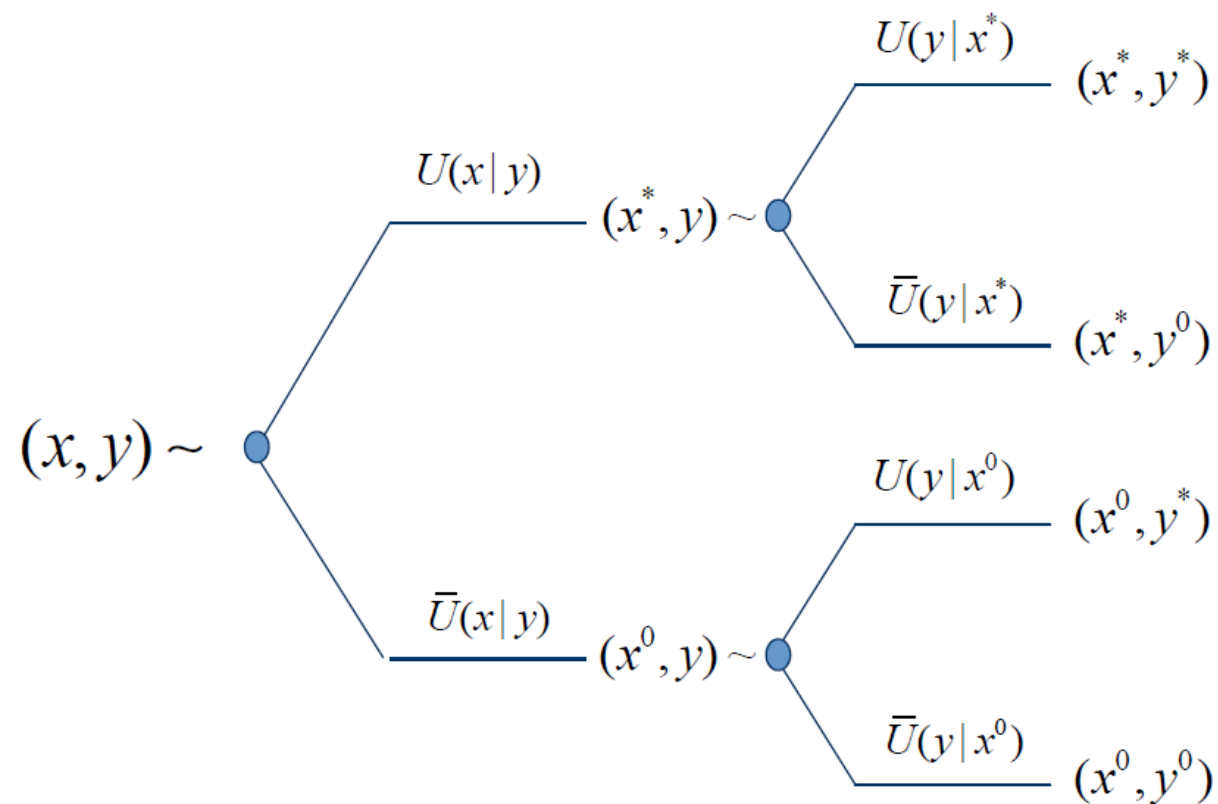
Simplified version can be obtained with additional assumptions:

Utility independence, $U(x|y) = U(x|\underline{y}) = U(x|\bar{y}) \quad \forall y$

Boundary independence, $U(x|\underline{y}) = U(x|\bar{y})$

Utility Trees and Bidirectional Utility Diagrams

$$U(x, y) = \underbrace{U(\bar{x}, y)}_{=U(y|\bar{x}) \cdot U(\bar{x}, \bar{y}) + [1 - U(y|\bar{x})] \cdot U(\bar{x}, \underline{y})} U(x|y) + \underbrace{U(\underline{x}, y)}_{=U(y|\underline{x}) \cdot U(\underline{x}, \bar{y}) + [1 - U(y|\underline{x})] \cdot U(\underline{x}, \underline{y})} [1 - U(x|y)]$$



Utility Trees and Bidirectional Utility Diagrams

... and one can define directional utility diagrams

$\boxed{x} \quad \boxed{y}$: mutual utility independence

$\boxed{x} \rightarrow \boxed{y}$: Directional utility independence, x independent of y

$\boxed{x} \leftarrow \boxed{y}$: Directional utility independence, y independent of x

$\boxed{x} \leftrightarrow \boxed{y}$: no independence

In higher dimension, it is more complex...

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