

# Multiattribute decision making

## Part 1: Classics (Fishburn/Kreps/Wakker)

### Separability and Independence

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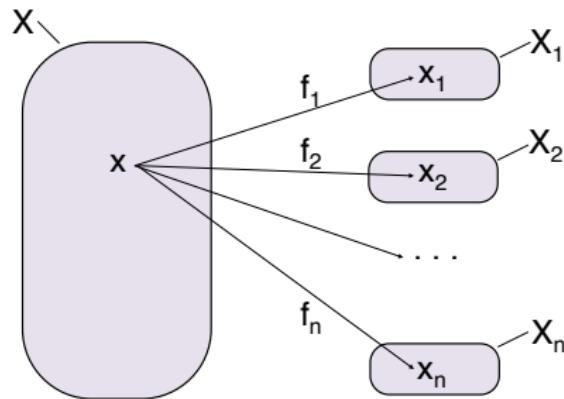
# Introduction

## Attribute mapping: Settings

- ▶  $X$  the set of decision alternatives
- ▶  $n \geq 2$  attributes used to differentiate among the objects in  $X$
- ▶  $\forall i \in 1 \dots n$ , there is a set  $X_i$  whose elements are values/level of attribute  $i$
- ▶  $\forall i \in 1 \dots n$ , there is an **attribute mapping**  $f_i : X \rightarrow X_i$ ; each  $f_i$  assigns to each object in  $X$  a specific level of the  $i$ th attribute.
- ▶ for each  $x \in X$ , the  $f_i$  functions produce a  $n$ -tuple  $(f_1(x), f_2(x), \dots, f_n(x))$  in  $X_1 \times \dots \times X_n$ .

## Attribute mapping: Settings

- ▶ The  $f_i$  functions describes  $x$  in terms of its values on the  $n$  attributes.
- ▶ The  $f_i$  functions are descriptive or identification functions.



$$x \rightarrow (f_1(x), f_2(x), \dots, f_n(x))$$

## Attribute mapping: Examples

$$f_1(x), f_2(x), \dots, f_n(x)$$

- ▶  $X$  is a set of simple probability measures on  $\mathbb{R}$  and  $f_i(x)$  is the  $i$ th central moment of  $x$ .
- ▶  $X$  is the set of commodity bundles. In standard microeconomics  $f_i$  are linear functions.
- ▶  $X$  is the set of commodity bundles. In Lancaster (1966) decision needs to map first each  $x \in X$  into a vector of characteristics  $(f_1(x), f_2(x), \dots, f_n(x))$ .
- ▶  $X$  is the set of people. In literature on inequality, decision is often based on a vector of wealths  $(f_1(x), f_2(x), \dots, f_n(x))$ .

## Attribute mapping: Remarks

$$x \rightarrow (x_1, x_2, \dots, x_n)$$

- ▶ In many cases,  $f_i(x)$  is abbreviated as  $x_i$  and elements in  $X$  are n-tuples  $(x_1, \dots, x_n)$  in the product set  $X_1 \times \dots \times X_n$ .
- ▶  $X$  is a proper subset of  $X_1 \times \dots \times X_n$ :  $X \subseteq X_1 \times \dots \times X_n$ .
- ▶ elements of  $X_1 \times \dots \times X_n$  that are not in  $X$  are unrealizable or infeasible combinations of attributes.
- ▶ many axiomatic preference theories assume that  $X = X_1 \times \dots \times X_n$ .
- ▶ implicit assumption of attribute mapping (non neutral): two elements in  $X$  that map in the same n-tuple  $(x_1, \dots, x_n)$  are indifferent/ have equal value.

## Criterion functions: Definition

- ▶ criterion function: real value function on  $X$  reflecting the value of the elements in  $X$ .
- ▶ various names: objective functions, goal functions, scoring functions, ranking functions, utility functions.
- ▶ difference between:
  - ▶ attribute mapping: objective characteristics of alternatives/consequences
  - ▶ criterion functions: subjective values on a scale.
- ▶ with  $m$  criteria and criteria functions  $g_j : X \rightarrow \mathbb{R}$ , each  $x$  is mapped into:
$$(g_1(x), \dots, g_m(x)) \tag{1}$$
- ▶ convention: preference monotonically increases in each  $g_j$ .

## Multiattribute structure of multiattribute choice models

- ▶ attribute mapping:  $x \rightarrow (f_1(x), f_2(x), \dots, f_n(x))$
- ▶ criterion function:  $(g_1(x), \dots, g_m(x))$
- ▶ composite multiattribute / multi-criteria mapping:

$$x \rightarrow (g_1(f_1(x), f_2(x), \dots, f_n(x)), \dots, g_m(f_1(x), f_2(x), \dots, f_n(x)))$$

$$x \rightarrow (g_1(x_1, x_2, \dots, x_n), \dots, g_m(x_1, x_2, \dots, x_n))$$

## Classification of theories: basic classification

- ▶ evaluation theories under certainty (Debreu, 1959)
- ▶ evaluation theories under risk (Keeney and Raiffa, 1976), based on expected utility
- ▶ evaluation theories under uncertainty (Savage, 1954)

## Classification of theories: 7 key aspects (Fishburn, 1978)

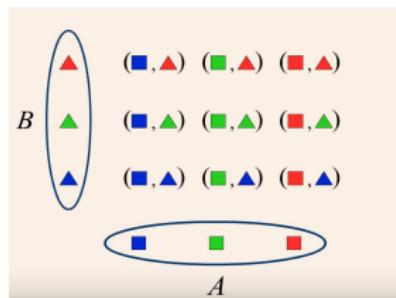
1. the number and nature of attributes and criterion functions
2. the structure of the feasible set of alternatives
3. the basis of evaluation
4. the ordering assumptions
5. **the independence assumptions**
6. the degree of compensatoriness
7. the subjective judgements of the decision-maker

## Aspect 1: the number and nature of attributes

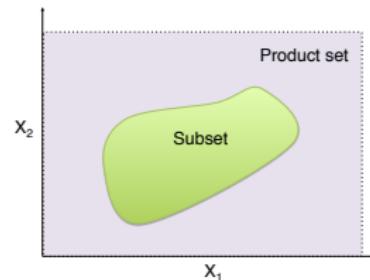
- ▶ The number of attributes:
  - ▶  $n$  finite, e.g.  $n = 2, n = 3, n > 3\dots$
  - ▶  $n = 2$  and  $n \geq 3$  can be very different
  - ▶  $n$  infinite, e.g. delayed outcomes
- ▶ The nature of attributes:
  - ▶  $X_i \in \{0, 1\}$  (qualitative)
  - ▶  $X_i \in \mathbb{R}$  (quantitative)

## Aspect 2: the structure of the feasible set

- ▶ Some theories assume that  $X$  or  $\{(g_1(x), \dots, g_m(x)) : x \in X\}$  is



(a) a Cartesian Product



(b) a Subset of product set

- ▶ feasible sets are convex/connected/compact separable topological or Euclidean spaces
- ▶ less structured feasible sets
- ▶  $X_i$  dense is much simpler, most tractable results for cartesian products

## Aspect 3: the basis of evaluation

- ▶ many theories are based on holistic binary preferences relation
- ▶ theories based on quaternary preference-intensity comparison relation
- ▶ revealed-preference theory
- ▶ stochastic preference theory
- ▶ most theories are based (or can be interpreted) on **binary comparisons**

## Aspect 4: ordering assumptions

- ▶ asymmetric preference relation:  $x \succ y$
- ▶ complete preference relation:  $x \succ y$  or  $y \succ x$  (or both)
- ▶ transitivity  $x \succ y$  and  $y \succ z \Rightarrow x \succ z$
- ▶ negative transitivity  $x \succ z \Rightarrow$  either  $x \succ y$  or  $y \succ z$
  
- asymmetric + transitive = strict partial order
- asymmetric + negatively transitive = strict weak order
  - ▶ the indifference relation  $\sim$  is an equivalence relation (reflexive, symmetric, transitive)
  - ▶ the relation  $\lesssim$  is a weak order (reflexive, complete, transitive)
- without completeness, an incomparability relation is assumed for  $\succ$  and  $\sim$
- transitivity is key

## Aspect 5: independence assumptions

Crucial element to obtain an additive representation of preferences  
(Gossen, 1854)

$$U(x) = \sum_i u_i(x_i)$$

- ▶ separability assumptions (strong, weak)
- ▶ independence assumptions
- ▶ widely used in (applied) economics
  - ▶ functional forms (Cobb-Douglas, CES, Bergson...)
  - ▶ applied and normative economics: EU, time preferences
  - ▶ decentralization and aggregation
  - ▶ utility trees, price/quantities indexes...

## Remarks on additivity

$$U(x) = \sum_i u_i(x_i)$$

- ▶ additivity implies independence of marginal utility of attribute  $i$  from the level of any other attribute.
- ▶ second order cross-partial derivatives equal to zero (e.g broad aggregates of goods: food, clothing, housing).
- ▶ nice property: take into account independence of certain attributes
- ▶ unfortunate property: defined in a cardinal way: public enemy number one for ordinalists
- ▶ solution:
  - ▶ reject the definition of additivity in terms of second order cross partial derivatives
  - ▶ define additivity in terms of the particular restrictions which it implies.

## Remarks on additivity in consumer theory: cross-price derivatives

- ▶ cross-price derivatives are proportional to income derivatives ( $\frac{\partial x_i}{\partial p_k} = \mu \frac{\partial x_i}{\partial w}$ , for  $i \neq k$ )
  - ▶ the change in the demand for food (clothing) induced by a change in the price of housing is proportional to the change in the demand for food (clothing) induced by a change in income.
  - ▶ the factor of proportionality  $\mu$  is invariant to the quantity of good  $i$  (but not to price  $p_k$ ).

$$\frac{\partial x_i}{\partial p_j} + x_j \frac{\partial x_i}{\partial w} = -\frac{\lambda}{\partial \lambda / \partial w} \frac{\partial x_i}{\partial w} \frac{\partial x_j}{\partial w}, i \neq j$$

## Remarks on additivity in consumer theory: substitution effects

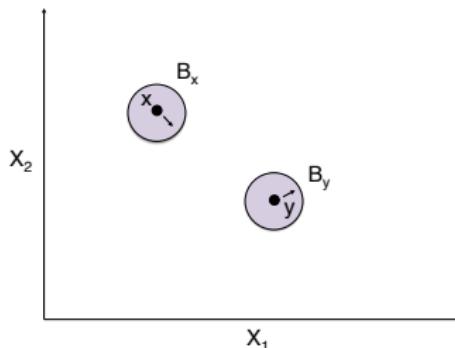
- ▶ substitution analyzed in terms of "competition for the consumer's dollars"
- ▶ substitution effect proportional to income derivative and Lagrange multiplier  $\lambda$
- ▶ Slutsky equation invariant to monotonic increasing transformations
  - all utility functions belonging to the same class as a given additive utility function have the same substitution effects
- ▶ but Lagrange multiplier is not invariant to any monotonic increasing transformations, only to linear one.
- ▶ well, I'm not totally sure of what happens here → needs deeper (basic) analysis
- ▶ all attributes are substitutes  
independent goods in cardinal sense are indeed substitutes in an ordinal sense

## Remarks on additivity in consumer theory (indirect utility)

- ▶ Houthakker (1960): additive indirect utility function
  - ▶ cross-price derivatives are proportional to the quantities affected.
  - ▶ all cross-price elasticities with respect to each price are the same
  - ▶ if both direct and indirect additive utilities are the same, all elasticities with respect to total expenditure are equal and unitary (straight Engel curves).

## Aspect 6: degree of compensatoriness

- ▶ example: local changes that preserve indifference can be made around one point in the space. Small deviations from  $x$  or  $y$  does not reverse the ranking.
- ▶ typical noncompensatory preferences: lexicographic preferences (preference ranking are reversed at the limit point of the sequence and indifference sets are singletons).
- ▶ key axiom: continuity or Archimedean axiom.
- ▶ variety of intermediary cases.



## Aspect 7: decision maker's subjective judgements

- ▶ some evaluative models impose the same evaluate realization to all decision makers.
  - ▶ monotonicity, efficient frontiers
  - ▶ first-order stochastic dominance
  - ▶ vector dominance relations if each criterion function is ordinaly equivalent across decision makers.
- ▶ most compensatory preference models presume different decision makers have different tradeoff structures.

## Evaluation theories without probabilities

## Basis references

- ▶ Debreu (1960), Gorman (1968): quantity aggregation.
- ▶ Blackorby, Primont, and Russell (1978): direct and indirect utility function, duality and expenditure function.
- ▶ Strotz (1957), Gorman (1959): price aggregation and two-stage budgeting.

## Building blocks

- ▶ no probabilities, no uncertain events
- ▶ set of objects to be evaluated is a subset  $X$  of a product set  $X_1 \times X_2 \times \dots \times X_n$
- ▶ evaluation with a global preference/utility function or one of the criterion functions  $g_j$  defined over  $X$
- ▶ asymmetric preference relation on  $X$  "better than":  $x \succ y$  and  $y \succ x$  cannot both hold,  $\forall x, y \in X$ .

## Separability in economics

- ▶ separability : reduces the complexity of the allocation decision
- ▶ decentralization of preferences:
  - ▶ first allocation to broad classes of commodities (e.g Food and Clothes "budgets")
  - ▶ then allocation within each class knowing only intra-class prices (e.g food prices)
  - ▶ allows to group items in close substitutes or complements
- ▶ Leontief(1947)-Sono(1945,1961) definition of separability:

$$\frac{\partial}{\partial x_k} \left( \frac{\partial U(x)/\partial x_i}{\partial U(x)/\partial x_j} \right) = 0$$

- ▶ MRS between pairs of goods in the first group (e.g food, including  $i, j$ ) is independent of quantities in the second group (e.g clothes, including  $k$ )
- ▶ an aggregator function exists for the first group

## Separability (Blackorby, Primont and Russel, 1978)

- ▶ binary partition of the set of the variables indices:

$$\left( \underbrace{x_1, \dots, x_i}_{\text{group 1: } x^1}, \underbrace{x_{i+1}, \dots, x_n}_{\text{group 2: } x^2} \right)$$

- ▶ conditional preference ordering over group 2:  $\succsim_{(x^1)}^2$ :

$$x^2 \succsim_{(x^1)}^2 \hat{x}^2 \Leftrightarrow (x^1, x^2) \succsim (x^1, \hat{x}^2)$$

- ▶ commodities 2 are separable from commodities 1 if  $\succsim_{(x^1)}^2$  is independent of the quantities in group 1: this defines  $\succsim^2$
- ▶ equivalent to the characterization:

$$U(x^1, x^2) \geq U(x^1, \hat{x}^2) \Leftrightarrow U(\hat{x}^1, x^2) \geq U(\hat{x}^1, \hat{x}^2)$$

## Separability and functional structure (Debreu, 1959, Gorman, 1968)

- ▶  $\succsim$  complete, continuous, transitive  $\Rightarrow$  same for  $\succsim^2$
- ▶ utility function for commodities in group 2:  $U^2$
- ▶  $U^2$  is independent of  $x^1$ :  $u_2 = U(x^2)$  group-2 utility
- ▶  $U^2$  is an aggregator function, for a given  $a^1$  in group 1:  
$$U^2(x^2) = U(a^1, x^2)$$
- ▶  $U^2$  aggregate quantities for group-2 commodities
- ▶ allows to define a macro function  $\mathcal{U}$  of  $x^1$  and the aggregator:

$$\mathcal{U}(x^1, U^2(x^2)) := U(x)$$

- ▶ not necessarily a symmetric concept, not necessarily unique
- ▶ as long as the allocation in group 2 is optimal, within-group (i.e group 2) allocation will be optimal

## Price aggregation (Gorman, 1959)

- ▶ drawback of separability: the consumer needs the optimal expenditure in each sector before proceeding to within-sector allocations
- ▶ allocation depends on all prices and income → decentralization is disappointing.
- ▶ idea: use sectoral prices and total expenditure as information rather than  $n$  commodity prices.
  - ▶ homogeneity of degree zero of the indirect utility function ensure the existence of price aggregation (no separability here)
  - ▶ means that demand for commodity  $i$  in sector  $r$  depends on income, vector of price indexes and prices.
  - ▶ optimal allocation in sector  $r$  based on income and price indexes, demand still depends on all prices (because of no separability)

## Two-stage budgeting (Strotz, 1957, Blackorby and Russel, 1997)

- ▶ separability (quantity aggregation) alone is disappointing, price aggregation alone is a bit useless...
- ▶ Strotz (1957): use both!
- ▶ question: under what conditions can the consumer
  - ▶ determine optimal group expenditure using only group price indices
  - ▶ spend the allocated budget using only sector-specific prices
- ▶ Preferences are separable in  $R$  groups and the conditional indirect utility function is twice-differentiable.
  - ▶ indirect utility function of the generalized Gorman polar form
  - ▶ aggregator function are generalization of homotheticity plus homothetic sectoral utility functions.
- ▶ Econometrics and identification: Blundell and Robin's (2000) latent separability

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## Additive structures

Basically, solve the dimensionality problem by dividing the products into smaller groups and allow for a flexible functional form within each group.

$$U(x) = \mathcal{U}(U^1(x^1) + U^2(x^2) + \dots + U^n(x_n)) := \sum_i^n u_i(x_i)$$

- ▶ one of oldest idea in the representation of preferences (Gossen, 1854)
- ▶ usually judged as too restrictive for economics, but....
  - ▶ Bergson-Samuelson, with  $u_i$  including different weights
  - ▶ Discounted utility model, with  $u_i$  including different time weights (constant discounting, hyperbolic discounting...).
  - ▶ Cobb-Douglas, quasi-linear, CES functions (e.g Spence-Dixit-Stiglitz)
  - ▶ Stone-Geary, almost ideal demand system, Pigou's law (Deaton, 1974)
  - ▶ decentralization and aggregation, price indexes...

## Additive structures and separability

- ▶ separability on subsets  $\{1, \dots, R\}$  allows for macro functions:  
$$U(x) = \mathcal{U}(u^1(x^1), u^2(x^2), \dots, u^R(x^R), x^{1\dots R^c})$$
- ▶ to go one step further: two versions of separability:
  - ▶ weak separability (on all coordinates)
  - ▶ strong separability (on all possible partitions)
- ▶ weak separability corresponds to  
$$U(x) = \mathcal{U}(u_1(x_1), u_2(x_2), \dots, u_n(x_n))$$
 for  $n \geq 3$ .
- ▶ strong separability corresponds to the additive representation for  $n \geq 3$ :

$$U(x) = \sum_i^n u_i(x_i)$$

- ▶ for  $n = 2$  further conditions are necessary (e.g the Thomsen condition).

## Separability over coordinates

- ▶  $X_i$  is separable from the other attributes/criteria if and only if:

$$\begin{aligned}(a_1, \dots, x_i, \dots, a_n) &\succsim (a_1, \dots, y_i, \dots, a_n) \\ \Rightarrow (b_1, \dots, x_i, \dots, b_n) &\succsim (b_1, \dots, y_i, \dots, b_n)\end{aligned}$$

for all cases where the four n-tuples are in  $X$ .

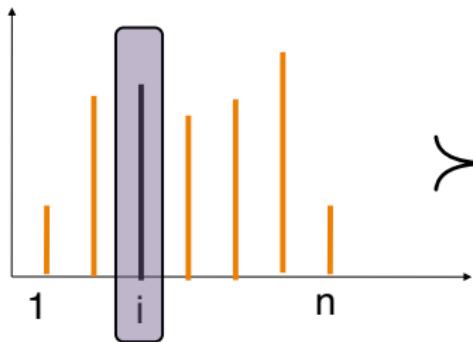
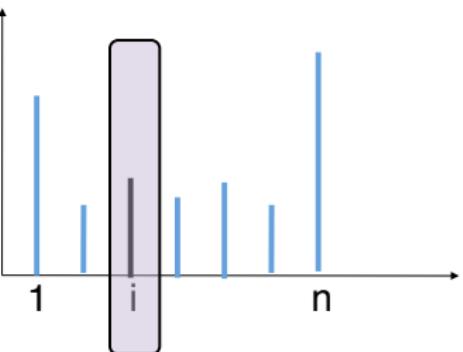
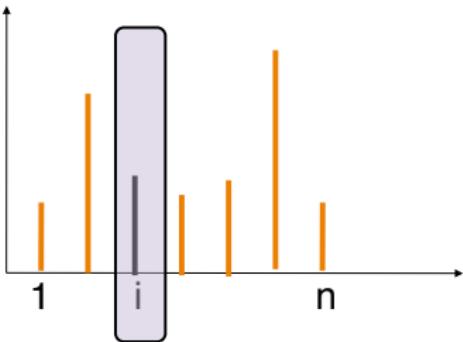
- ▶ allows to define an asymmetric relation  $\succsim_i$  on  $X_i$  from  $\succsim$  on  $X$  by:

$$x_i \succsim_i y_i \Leftrightarrow (a_1, \dots, x_i, \dots, a_n) \succsim (a_1, \dots, y_i, \dots, a_n)$$

for some  $(a_1, \dots, x_i, \dots, a_n), (a_1, \dots, y_i, \dots, a_n) \in X$

- ▶  $X$  is supposed to be a large subset of  $X_1 \times \dots \times X_n$  to avoid triviality: cases where  $\succsim_i$  is empty, i.e there are never two n-tuples that have the same values of  $X_i$  for all but one  $i$  and different values of  $X_i$  for the other  $i$ .

## Weak separability for $\succ$ on $X$

 $\succ$  $\succ$

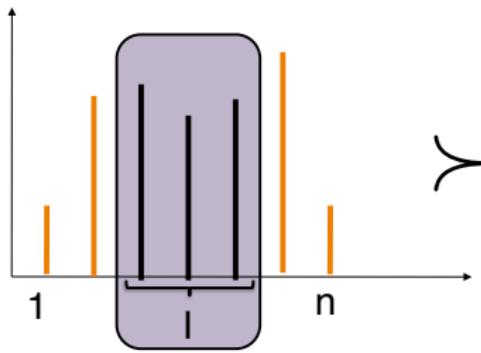
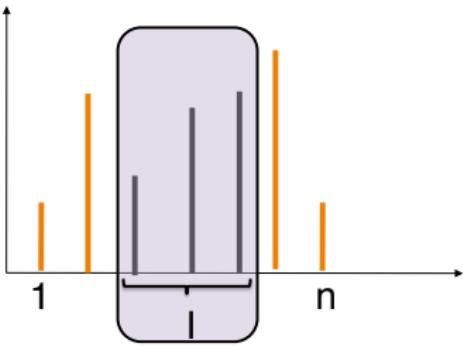
## Strong separability

- ▶ a subset  $I : \{X_i : i \in I\}$  is strongly separable from its complement  $I^c : \{1, \dots, n\} \setminus I$  if and only if:
$$(x_i \text{ for } i \in I, a_i \text{ for } i \in I^c) \succ (y_i \text{ for } i \in I, a_i \text{ for } i \in I^c)$$
$$\Rightarrow (x_i \text{ for } i \in I, b_i \text{ for } i \in I^c) \succ (y_i \text{ for } i \in I, b_i \text{ for } i \in I^c)$$

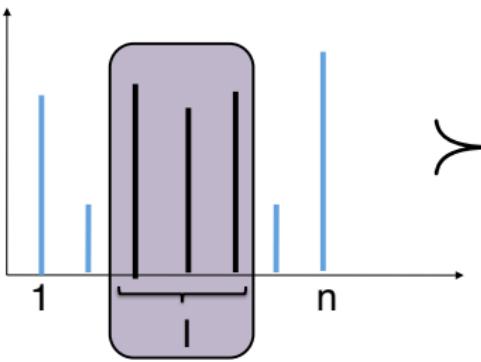
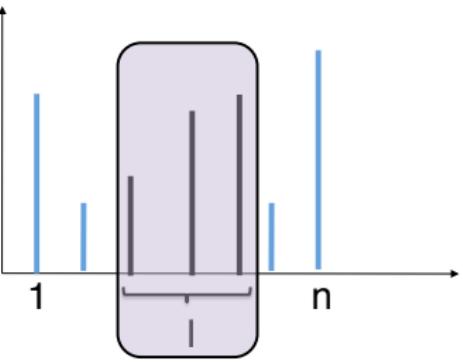
for all cases where the four n-tuples are in  $X$ .

- ▶ typical exemple (Kreps):  $I$  are food items,  $I^c$  are clothing, housing... Separability means:
  - ▶ that combinations of food does not depend whether the consumers are wearing skirts or t-shirts

## Strong separability for $\succ$ on $X$

 $\succ$ 

1

 $n$  $\succ$ 

# Independence

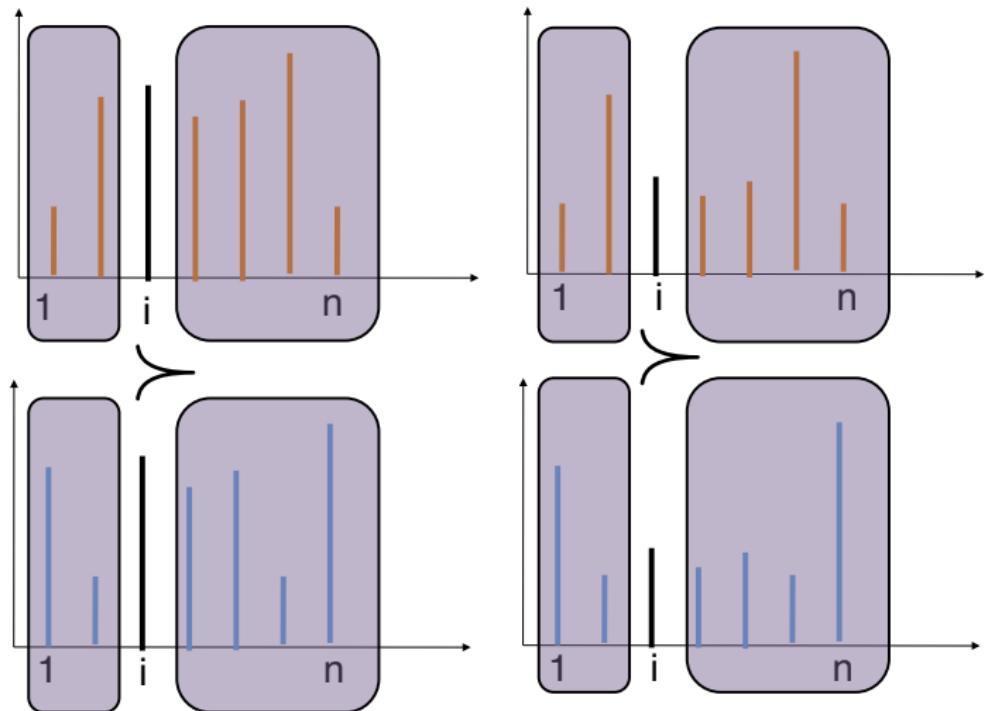
- ▶ Coordinate independence:

$$\begin{aligned}(x_1, \dots, a_i, \dots, x_n) &\succsim (y_1, \dots, a_i, \dots, y_n) \\ \Leftrightarrow (x_1, \dots, b_i, \dots, x_n) &\succsim (y_1, \dots, b_i, \dots, y_n)\end{aligned}$$

for all *i* cases where the four n-tuples are in  $X$ .

- ▶ Coordinate independence implies weak separability
- ▶ Coordinate independence allows to replace any single pair of equal coordinates
- ▶ Weak separability allows to replace  $n - 1$  t-uples of pairs of equal coordinates
- ▶ Notation  $a_i x \succsim a_i y \Leftrightarrow b_i x \succsim b_i y$
- ▶ if all the  $X_i$ 's are the same, the condition is called the "sure-thing principle".

## Coordinate independence



## Independence and additive utility (Krantz et al. 1971)

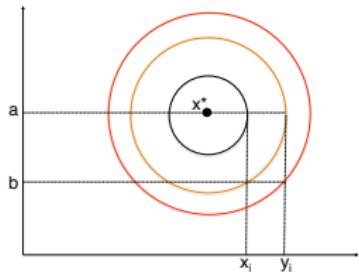
- ▶ outcome set  $X = \prod_{i=1}^n X_i$  with  $n \geq 3$
- ▶  $\sim$  binary relation on  $X \times X$  satisfying
  - ▶ essentiality (each attribute matters)
  - ▶ solvability wrt. every attribute
- ▶  $\sim$  weak order + independence and Archimedean axiom for every attribute

## Limitations

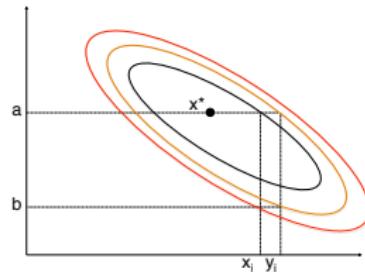
- ▶ not general  $n = 2$  needs additional assumptions
- ▶  $X$  is a Cartesian product of the  $X_i$ , if not additional- highly technical - assumption needed (e.g Chateauneuf and Wakker, 1993)
- ▶ restricted solvability has only meaning for continuums (time, money) but not for discrete sets (qualitative)

# Interdependent preferences

- ▶ Debreu (1960): independence for some subsets of goods may fail when other subsets are independent of their complements.
- ▶ Multiplicative form  $u(x_1, \dots, x_n) = u_1(x_1) \times \dots \times u_n(x_n)$  without constant sign:  $u_i(x_i) > 0$  and  $u_i(y_i) < 0$ .
- ▶ ideal points  $x^*$ : independence vs. interdependence:



(c) spherical isouility



(d) elliptical isouility

## Next time...

- ▶ interpretation of the axiomatic in terms of indifference curves
- ▶ elicitation of utility functions for multiattribute choice
- ▶ standard sequence and tradeoff consistency
- ▶ risk and uncertainty